

k - $L(2, 1)$ -Labelling for Planar Graphs is NP-Complete for $k \geq 4$

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Abstract

A mapping from the vertex set of a graph $G = (V, E)$ into an interval of integers $\{0, \dots, k\}$ is an $L(2, 1)$ -labelling of G of span k if any two adjacent vertices are mapped onto integers that are at least 2 apart, and every two vertices with a common neighbour are mapped onto distinct integers. It is known that for any fixed $k \geq 4$, deciding the existence of such a labelling is an NP-complete problem while it is polynomial for $k \leq 3$. For even $k \geq 8$, it remains NP-complete when restricted to planar graphs. In this paper, we show that it remains NP-complete for any $k \geq 4$ by reduction from Planar Cubic Two-Colourable Perfect Matching. Schaefer stated without proof that Planar Cubic Two-Colourable Perfect Matching is NP-complete. In this paper we give a proof of this.

1 Introduction

The Frequency Assignment Problem requires the assignment of frequencies to radio transmitters in a broadcasting network with the aim of avoiding undesired interference and minimising bandwidth. One of the longstanding graph theoretical models of this problem is the notion of distance constrained labelling of graphs. An $L(2, 1)$ -labelling of a graph G is a mapping from the vertex set of G into the nonnegative integers such that the labels assigned to adjacent vertices differ by at least 2, and labels assigned to vertices at distance 2 are different. The *span* of such a labelling is the maximum label used. In this model, the vertices of G represent the transmitters and the edges of G express which pairs of transmitters are too close to each other so that undesired interference may occur, even if the frequencies assigned to them differ by 1. This model was introduced by Roberts [12] and since then the concept has been intensively studied (see the survey of Yeh [14]).

In their seminal paper, Griggs and Yeh [8] proved that determining the minimum span of a graph G , denoted $\lambda_{2,1}(G)$, is an NP-hard problem. Fiala et al. [5] proved that deciding $\lambda_{2,1}(G) \leq k$ is NP-complete for every fixed $k \geq 4$ and later Havet and Thomassé [9] proved that for any $k \geq 4$, it remains NP-complete when restricted to bipartite graphs (and even a restricted family of bipartite graphs, i.e. *incidence graphs* or *first division of graphs*). When the span k is part of the input, the problem is nontrivial for trees but a polynomial time algorithm based on bipartite matching was presented in [3]. The problem is still solvable in polynomial time if the input graph is outerplanar [10, 11].

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Moreover, somewhat surprisingly, the problem becomes NP-complete for series-parallel graphs [4], and thus the $L(2, 1)$ -labelling problem belongs to a handful of problems known to separate graphs of tree-width 1 and 2 by P/NP-completeness dichotomy.

In this paper we consider the following problem.

Problem 1 (Planar k - $L(2, 1)$ -Labelling).

Let $k \geq 4$ be fixed.

Instance: A planar graph G .

Question: Is there an $L(2, 1)$ -labelling with span k ?

Bodlaender *et al.* [1] showed that this problem is NP-complete if we require $k \geq 8$ and k even. In the survey paper [2], it is suggested that the problem is NP-complete for all $k \geq 8$ due to [6]. However this does not seem to be the case. In [6] there is a proof showing that the corresponding problem where k is specified as part of the input is NP-complete. This proof shows that the problem is NP-complete for certain fixed values of k . However it is far from clear for which values of k this is true. The same authors also show in [7] that the problem is NP-complete for $k = 8$.

In this paper we first prove that Planar Cubic Two-Colourable Perfect Matching, which we define in the next section, is NP-Complete. This result was first stated by Schaefer [13] but without proof. In the second part of this paper we use this result in order to show that Problem 1 is NP-complete.

2 Preliminary results

The starting problem for our reductions is Not-All-Equal 3SAT, which is defined as follows [13].

Definition 2 (NOT-ALL-EQUAL 3SAT).

Instance: A set of clauses each having three literals.

Question: Can the literals be assigned value true or false so that each clause has at least one true and at least one false literal?

In [13], it is shown that this problem is NP-complete.

Our reduction involves an intermediate problem concerning a special form of two-colouring. In this section we define the intermediate problem and show that it is NP-complete. When $k = 4$ or $k = 5$, the final stage of our reduction is similar to the reduction in [5]. However we cannot use induction for higher values of k in contrast with the situation in [5] and the problem from which the reduction starts in [5] is not known to be NP-complete for planar graphs. So considerably more work is required.

The following problem is also discussed in [13].

Problem 3 (TWO-COLOURABLE PERFECT MATCHING).

Instance: A graph G .

Question: Is there a colouring of the vertices of G with colours black and white in which every vertex has exactly one neighbour of the same colour?

In [13] it was shown that Two-Colourable Perfect Matching is NP-complete. We are more interested in the case where the input is restricted to being a planar cubic graph. We call this variant, Planar Cubic Two-Colourable Perfect Matching defined formally as follows [13].

Problem 4 (PLANAR CUBIC TWO-COLOURABLE PERFECT MATCHING).

Instance: A planar cubic graph G .

Question: Is there a colouring of the vertices of G with colours black and white in which every vertex has exactly one neighbour of the same colour?

Schaefer [13] states that this problem is NP-complete but does not give the details of the proof. We call a colouring as required in Problem 4 a *two-coloured perfect matching*. This section is devoted to the proof of this result, using a reduction from Not-All-Equal 3SAT [13]. As far as we know, no proof of this has ever been published.

We say that a colouring of the vertices of a graph with colours black and white is an *almost two-coloured perfect matching* if every vertex of degree at least two is adjacent to exactly one vertex of the same colour. We say an edge is *monochromatic* if both endpoints have the same colour and *dichromatic* if its endpoints have different colours.

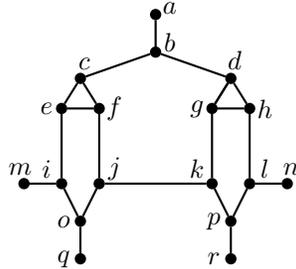


Figure 1: Planar graph H .

Let H be the planar graph depicted in Fig. 1. H plays a key role in showing that Problem 4 is NP-complete. We need the following lemma.

Lemma 5. *Any almost two-coloured perfect matching of H has the following properties.*

- *Exactly one of the edges ab, mi, ln is monochromatic.*
- *Vertices b, i, l receive the same colour.*
- *Vertices o, p, q, r receive the other colour to b, i, l .*

Proof. Consider the triangles on the vertices c, e, f and g, h, d . In order to obtain an almost two-coloured perfect matching exactly one of the edges ce, ef and cf must be monochromatic. The same is true for the triangle on the vertices g, h, d . Now consider the subgraph of H induced by the vertices $c, e, f, i, j, m, k, o, q$. In Fig. 2 three of the six almost two-coloured perfect matchings of this subgraph are depicted with monochromatic edges shown by heavy lines. The other three two-coloured perfect matchings are obtained by interchanging the colours. It follows that oq and pr must be monochromatic.

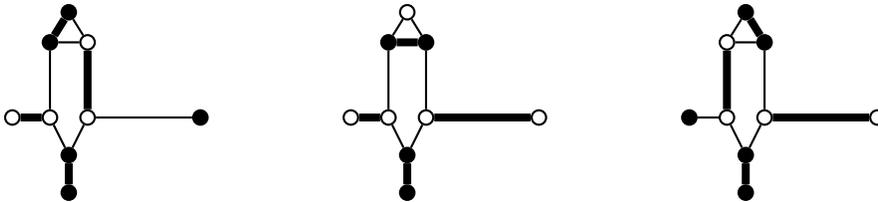


Figure 2: Three almost two-coloured perfect matchings of the subgraph of H induced by the vertices $c, e, f, m, i, j, k, o, q$.

By symmetry the same applies to the subgraph of H induced by the vertices $d, g, h, j, k, l, n, p, r$. Considering which pairs of these almost two-coloured perfect matchings are compatible and extend to an almost two-coloured perfect matching of H shows that there are only six possibilities. In Fig. 3 three possible almost two-coloured perfect

matchings are depicted. The only other possible almost two-coloured perfect matchings are obtained by interchanging the two colours. Clearly these all have the properties described in the lemma.

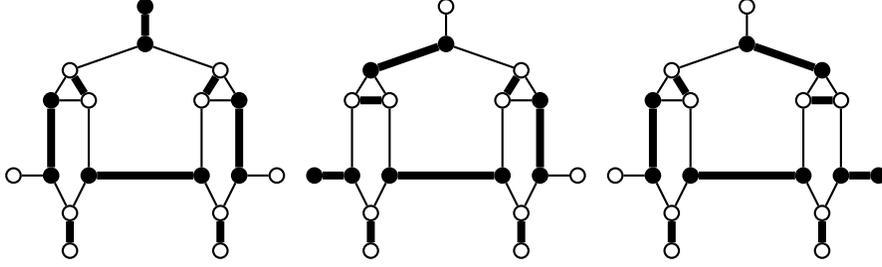


Figure 3: Almost two-coloured perfect matchings of H .

□

We define what we call the *clause gadget graph* K as follows, see Fig. 4. Take three copies of H , namely H_1 , H_2 and H_3 . We label the vertices by adding the subscript $i \in \{1, 2, 3\}$ to the corresponding label of H . Now identify a_1, a_2, a_3 into a single vertex a , remove vertices $m_1, m_2, m_3, n_1, n_2, n_3$ and their incident edges and replace them with edges l_1i_2, l_2i_3, l_3i_1 . Notice that K is planar and every vertex has degree three, except for q_1, q_2, q_3 and r_1, r_2, r_3 .

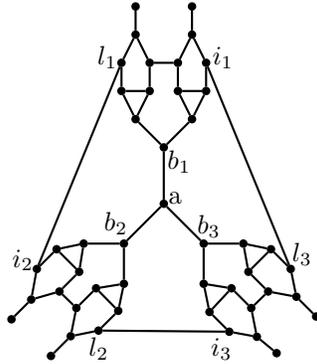


Figure 4: Planar clause gadget K .

Lemma 6. *A two-colouring of $\bigcup_{t=1}^3 \{o_t, q_t, p_t, r_t\} \cup \{a\}$ may be extended to an almost two-coloured perfect matching of K if and only if*

- For each $t = 1, 2, 3$, o_t, q_t, p_t, r_t are monochromatic and o_t, p_t, q_t, r_t all receive the same colour.
- For exactly two values of $t = 1, 2, 3$, the vertices o_t, p_t, q_t, r_t receive the same colour as a .

Proof. We first show that any almost two-coloured perfect matching of K must have the two properties in the lemma.

The first property is an immediate consequence of Lemma 5.

To show that the second property holds, recall that exactly one neighbour of a must receive the same colour as a . Let b_{t_1} for $1 \leq t_1 \leq 3$ be this neighbour. Then from

Lemma 5 we know that b_{t_1} must have the opposite colour to $o_{t_1}, p_{t_1}, q_{t_1}, r_{t_1}$. Since the other neighbours of a , namely b_{t_2} and b_{t_3} for $t_2, t_3 \in \{1, 2, 3\} \setminus \{t_1\}$, receive the opposite colour to a , the vertices $o_{t_2}, p_{t_2}, q_{t_2}, r_{t_2}, o_{t_3}, p_{t_3}, q_{t_3}, r_{t_3}$ must receive the same colour as a .

Now we show that any two-colouring of $\bigcup_{t=1}^3 \{o_t, q_t, p_t, r_t, b_t\} \cup \{a\}$ satisfying the conditions of the lemma may be extended to an almost two-coloured perfect matching of K . Suppose without loss of generality, a is coloured black and o_1, p_1, q_1, r_1 are coloured white. Then colour l_1, i_1 black and l_2, i_2, l_3, i_3 white. This colouring may be extended to an almost two-coloured perfect matching using the colourings of Fig. 3 and the colourings obtained from those in Fig. 3 by interchanging the colours. \square

We now move a step towards the main result of this section with the following proposition.

Proposition 7. *Problem 3 is NP-complete if the input is restricted to cubic graphs.*

Proof. Given an instance of Not-All-Equal 3SAT with clauses C_1, \dots, C_m , construct a graph as follows. For every clause C take a copy of the clause gadget graph K and do the following. Suppose without loss of generality C has literals x_1, x_2, x_3 . Label the two vertices of degree one of the subgraph H_i of $K(C)$ and their neighbours in H_i with x_i .

Now for each literal x do the following. Suppose that literal x appears in clauses C_{i_1}, \dots, C_{i_k} . (If x appears twice or three times in a clause C_r then add C_r twice or three times to this list.) For every $j = 1, \dots, k - 1$ remove either one of the vertices of degree one labelled x from $K(C_{i_j})$ and from $K(C_{i_{j+1}})$ leaving two half-edges. Now identify these two half edges to form an edge joining $K(C_{i_j})$ and $K(C_{i_{j+1}})$. Finally do the same thing with the remaining two edges labelled x in C_{i_k} and C_{i_1} . We call the graph obtained G .

Now suppose that there is a solution S of the instance of Not-All-Equal 3SAT given. For all literals x in Not-All-Equal 3SAT, colour all vertices in G that are labelled x with colour white if $S(x)$ is true and black if $S(x)$ is false. We now show that this colouring can be extended to a two-coloured perfect matching of G . First note that all edges joining copies of K are monochromatic since their endpoints are labelled with the same literal. In the next step colour the vertex a in each copy of K so that it has the same colour as the vertices o_t, p_t for exactly two values of $t = 1, 2, 3$. This is possible because for $t = 1, 2, 3$, o_t, p_t are labelled with literals x_1, x_2, x_3 which cannot have all the same value as they belong to one clause. By Lemma 6 we can extend the colouring of each copy of K to an almost two-coloured perfect matching of K which yields a two-coloured perfect matching of G .

Now suppose there is a two-coloured perfect matching of G . By Lemma 6 the edges joining the copies of K must be monochromatic. All vertices in G labelled with the same literal therefore must have the same colour and in each copy of K , for exactly two values of $t = 1, 2, 3$, the vertices o_t, p_t receive the same colour. It follows that if we assign to each literal x the value true if it is the label of white vertices and false if it is the label of black vertices then we obtain a solution to Not-All-Equal 3SAT. \square

We call the edges joining copies of K *identifying edges*.

In order to prove that Problem 4 is NP-complete we still need to deal with edges that cross. For this reason we define the *uncrossing gadget* U to be the graph depicted in Fig. 5.

Lemma 8. *There exists an almost two-coloured perfect matching of U if and only if w, v, z_1, z_2 have the same colour and a, b, z_3, z_4 have the same colour.*

Proof. Consider the four-cycle on the vertices c, e, f, d in U . In a two-coloured perfect matching exactly two of the vertices c, e, f, d must receive the colour black and the other two must receive the colour white. There are two different ways of colouring them. The first way is that exactly two of the edges in the four-cycle are monochromatic, namely ce

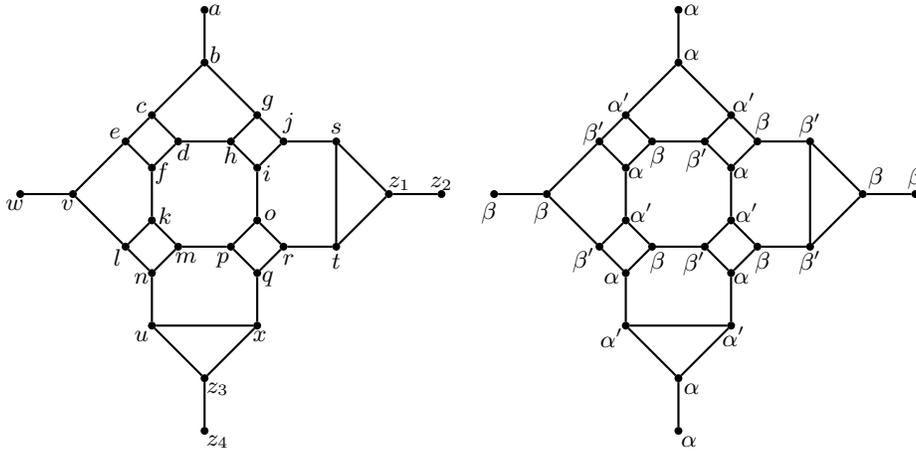


Figure 5: Uncrossing gadget U and its possible almost two-coloured perfect matchings where $\alpha, \beta \in \{\text{black, white}\}$ and α', β' are the opposite colours to α and β , respectively.

and df , or cd and ef . Then none of the edges cb, dh, ev, fk can be monochromatic. The other way is that none of the edges in the four-cycle is monochromatic and all of the edges cb, dh, ev, fk are monochromatic. The analogous thing is true for any four-cycle in U .

We now prove that in any almost two-coloured perfect matching the edges ab, wv, z_1z_2 and z_3z_4 are monochromatic. Suppose ab is dichromatic. Then precisely one of bc and bg must be monochromatic. Without loss of generality assume bc is monochromatic. Then the four-cycle on c, e, f, d cannot have a monochromatic edge. It follows dh must be monochromatic. But then bg must be monochromatic which is a contradiction. Thus ab and by symmetry wv must be monochromatic. Now suppose z_1z_2 is dichromatic. It follows that precisely one of sz_1 and tz_1 is monochromatic. Without loss of generality assume tz_1 is monochromatic. Then js must be monochromatic. It follows that io must be monochromatic and so must rt . This is not possible. Thus z_1z_2 and due to symmetry z_3z_4 must be monochromatic. Hence each of the four-cycles $(c, d, f, e), (g, h, i, j), (k, l, n, m), (o, p, q, r)$ contains exactly two monochromatic edges. So vertices that are opposite of each other in these four-cycles receive opposite colours. It is now easy to see that the only possible colourings are as shown in Fig. 5 where $\alpha, \beta \in \{\text{black, white}\}$ and α' denotes the opposite colour to α and β' denotes the opposite colour to β . The result then follows. \square

We are now able to prove that Problem 4 is NP-complete.

Theorem 9. *Problem 4 is NP-complete.*

Proof. Given an instance of Not-All-Equal 3SAT, construct the graph G as in the proof of Proposition 7. This graph can be drawn in the plane so that the only edges that cross are the identifying edges, each pair of identifying edges crosses at most once and at most two edges cross at any point.

Now we replace the crossings one by one by replacing a pair of crossing edges by the uncrossing gadget. Suppose γ and δ are two edges that cross. We delete γ and δ and replace them with a copy of the uncrossing gadget attaching w and z_2 to the endpoints of γ , and a and z_4 to the endpoints of δ . We will also call the four pendant edges wv, ab, z_1z_2, z_3z_4 in the uncrossing graph identifying edges. After each replacement we can draw the graph so that only identifying edges cross and such that there is one fewer crossing. We continue until there are no more crossing edges. The final graph can

be constructed in polynomial time and is planar and cubic. Each original identifying edge in G now corresponds to one or more identifying edges with each consecutive pair being on opposite sides of a copy of the uncrossing gadget. Lemma 8 shows that in a two-coloured perfect matching all of these edges must be monochromatic and all the endpoints of these edges have the same colour.

Now the argument in Proposition 7 shows that the final graph has a two-coloured perfect matching if and only if the instance of Not-All-Equal 3SAT is satisfiable. \square

3 k - $L(2, 1)$ -labelling for $k \geq 4$ fixed

Let G be a planar cubic graph. In order to establish our main result we will reduce Planar Cubic Two-Colourable Perfect Matching to Planar k - $L(2, 1)$ -Labelling for planar graphs for each $k \geq 4$. From any planar cubic graph G forming an instance of Planar Cubic Two-Colourable Perfect Matching, we construct a graph K . As we see in the next section, the basic form of K does not depend on k but is formed by constructing an auxiliary graph H and then replacing each edge of H by a gadget which does depend on k . In this section we define these gadgets and analyse certain $L(2, 1)$ -labellings of them. Each gadget has two distinguished vertices, which will always be labelled u and v , corresponding to the endpoints of the edge that is replaced in the auxiliary graph defined in the next section. These two vertices have degree $k - 1$ in K . Any vertex of degree $k - 1$ must receive either label 0 or k in a k - $L(2, 1)$ -labelling because these are the only possible labels for which there are $k - 1$ labels remaining to label the neighbours of that vertex, so we will analyse the k - $L(2, 1)$ -labellings of these gadgets in which u, v receive label 0 or k .

3.1 $\lambda_{2,1}(G) = 4$

In this subsection the gadget G_4 is simply a path of length three. More precisely the gadget G_4 is given by $V(G_4) = \{u, a_u, a_v, v\}$ and $E(G_4) = \{ua_u, a_u a_v, a_v v\}$. This gadget is used in [5], from where we get the following lemma.

Lemma 10. *There is a 4- $L(2, 1)$ -labelling L of G_4 with $L(u), L(v) \in \{0, 4\}$ if and only if the following conditions are satisfied.*

1. If $(L(u), L(v)) = (0, 0)$, then $(L(a_u), L(a_v)) \in \{(2, 4), (4, 2)\}$.
2. If $(L(u), L(v)) = (4, 4)$, then $(L(a_u), L(a_v)) \in \{(0, 2), (2, 0)\}$.
3. If $(L(u), L(v)) = (4, 0)$, then $(L(a_u), L(a_v)) = (1, 3)$.
4. If $(L(u), L(v)) = (0, 4)$, then $(L(a_u), L(a_v)) = (3, 1)$.

3.2 $\lambda_{2,1}(G) = 5$

Let G_5 be the graph depicted in Fig. 6.

Lemma 11. *There is a 5- $L(2, 1)$ -labelling L of G_5 with $L(u), L(v) \in \{0, 5\}$ if and only if the following conditions are satisfied.*

1. If $(L(u), L(v)) = (0, 0)$, then $(L(a_u), L(a_v)) \in \{(2, 5), (5, 2), (3, 5), (5, 3)\}$.
2. If $(L(u), L(v)) = (5, 5)$, then $(L(a_u), L(a_v)) \in \{(3, 0), (0, 3), (2, 0), (0, 2)\}$.
3. If $(L(u), L(v)) = (0, 5)$, then $(L(a_u), L(a_v)) = (4, 1)$.
4. If $(L(u), L(v)) = (5, 0)$, then $(L(a_u), L(a_v)) = (1, 4)$.

Proof.

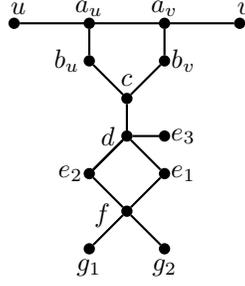


Figure 6: The edge gadget G_5 .

1. By the definition of $L(2, 1)$ -labelling, $L(a_u)$ and $L(a_v)$ are both in $\{2, 3, 4, 5\}$. As $|L(a_u) - L(a_v)| \geq 2$, $(L(a_u), L(a_v)) \in \{(2, 4), (4, 2), (2, 5), (5, 2), (3, 5), (5, 3)\}$.
Suppose for a contradiction that $(L(a_u), L(a_v)) \in \{(2, 4), (4, 2)\}$. By symmetry, we may assume that $(L(a_u), L(a_v)) = (2, 4)$. Then $L(b_u) = 5$ and $L(b_v) = 1$. The vertices d and f have degree four and thus must receive labels from $\{0, 5\}$. Because $\text{dist}(b_u, d) = 2$ we must have $L(d) = 0$ and because $\text{dist}(d, f) = 2$ we must have $L(f) = 5$. This implies that $\{L(e_1), L(e_2)\} = \{2, 3\}$. But then vertex c cannot be labelled, giving a contradiction. An $L(2, 1)$ -labelling is obtained if $(L(a_u), L(a_v)) = (5, 2)$ and $L(b_u) = 3$, $L(b_v) = 4$, $L(c) = 0$, $L(d) = 5$, $L(e_3) = 1$, $L(e_2) = 2$, $L(e_1) = 3$, $L(f) = 0$, $L(g_1) = 5$ and $L(g_2) = 4$ or if $(L(a_u), L(a_v)) = (5, 3)$ and $L(b_u) = 2$, $L(b_v) = 1$, $L(c) = 4$, $L(d) = 0$, $L(e_3) = 5$, $L(e_2) = 2$, $L(e_1) = 3$, $L(f) = 5$, $L(g_1) = 0$ and $L(g_2) = 1$. The other cases follow by symmetry.
2. Analogous to (i).
3. By the definition of $L(2, 1)$ -labelling, $L(a_u) \in \{2, 3, 4\}$ and $L(a_v) \in \{1, 2, 3\}$. As $|L(a_u) - L(a_v)| \geq 2$, $(L(a_u), L(a_v)) \in \{(4, 1), (4, 2), (3, 1)\}$. Suppose for a contradiction that $(L(a_u), L(a_v)) \neq (4, 1)$. By the label symmetry $x \mapsto 5 - x$, we may assume that $(L(a_u), L(a_v)) = (4, 2)$. Hence $L(b_u) = 1$ and $L(b_v) = 0$. Now the vertices d and f have degree four and thus $L(d) = 0, L(f) = 5$ or $L(d) = 5, L(f) = 0$. As $\text{dist}(b_v, d) = 2$, $L(d) = 5$ and as $\text{dist}(d, f) = 2$, $L(f) = 0$. Now $\{L(e_1), L(e_2)\} = \{2, 3\}$. But then vertex c cannot be labelled, a contradiction. An $L(2, 1)$ -labelling is obtained if $(L(a_u), L(a_v)) = (4, 1)$ and $L(b_u) = 2$, $L(b_v) = 3$, $L(c) = 0$, $L(d) = 5$, $L(e_3) = 1$, $L(e_2) = 2$, $L(e_1) = 3$, $L(f) = 0$, $L(g_1) = 5$ and $L(g_2) = 4$.
4. Analogous to (iii).

□

3.3 $\lambda_{2,1}(G) \geq 6$

In this subsection we introduce for any $k \geq 6$ the gadget G_k and consider some of its $L(2, 1)$ -labellings. The gadgets G_6 and G_7 are depicted in Fig. 7 and in Fig. 8, respectively.

Let H' be defined as follows, see Fig. 9.

$$V(H') = \{c, d, e, f_1, \dots, f_{k-3}, g, h, i\}$$

and

$$E(H') = \{cd, de, hg, gi, gf_1, \dots, gf_{k-3}, df_1, \dots, df_{k-3}\}.$$

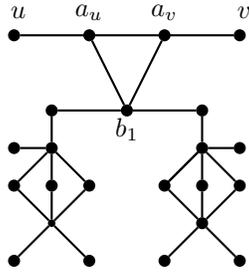


Figure 7: The edge gadget G_6 .

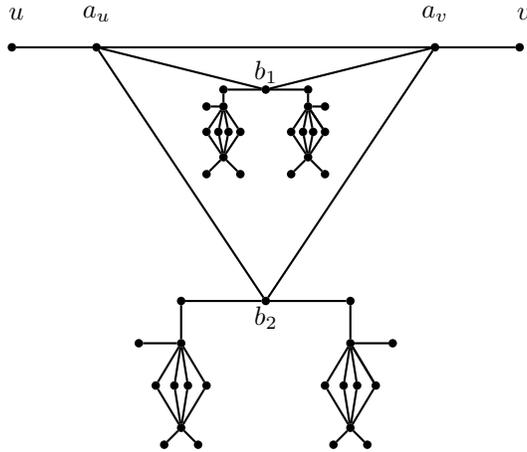


Figure 8: The edge gadget G_7 .

Lemma 12. *For any $k \geq 6$ the graph H' is planar and there exists an $L(2, 1)$ -labelling L of H' with span k if and only if*

$$(L(c), L(d)) \in \{(0, k), (1, k), (k-1, 0), (k, 0)\}.$$

Proof. As seen from Fig. 9, H' is planar for $k = 6$. For any higher k we need to connect $k - 6$ paths of length two at d and g to the graph H' in Fig. 9. So H' is planar for any $k \geq 6$.

If L is a k - $L(2, 1)$ -labelling of H' then $\{L(g), L(d)\} = \{0, k\}$ because g and d have degree $k - 1$ and $\text{dist}(g, d) = 2$. It follows that $\{f_1, \dots, f_{k-3}\} = \{2, \dots, k - 2\}$. Suppose that $L(d) = 0$ and $L(g) = k$. Then $\{L(h), L(i)\} = \{1, 0\}$ and $\{L(c), L(e)\} = \{k - 1, k\}$. Similarly if $L(d) = k$ and $L(g) = 0$, then $\{L(h), L(i)\} = \{k - 1, k\}$ and $\{L(c), L(e)\} = \{0, 1\}$. \square

We now define the graph G_k for $k \geq 6$. Take a path of length three with vertices u, u_a, a_v, v and edges $ua_u, a_u a_v, a_v v$. Add vertices b_1, \dots, b_{k-5} , with each joined to a_u and a_v . Now for each $i = 1, \dots, k - 5$, add two copies of H' with the vertex labelled c in each copy joined to b_i . So each of b_1, \dots, b_{k-5} has degree four. To refer to the vertices in the two copies of H' attached to b_i , we add a subscript of (l, i) to the name of the vertices in one copy of H' and (r, i) to the name of the vertices in the other copy of H' . Notice that G_k is planar.

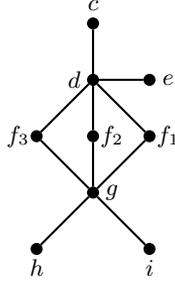


Figure 9: The graph H' for $k = 6$.

Lemma 13. *There exists a k - $L(2, 1)$ -labelling L of G_k with $L(u) = L(v) = 0$ if and only if $(L(a_v), L(a_u)) \in \{(2, k), (k, 2), (k-2, k), (k, k-2)\}$.*

Proof. First notice that we need $k-5$ different colours to colour the vertices b_1, \dots, b_{k-5} as they are all at distance two from each other. We first show that there exists a k - $L(2, 1)$ -labelling L of G_k with $L(u) = L(v) = 0$ and $(L(a_u), L(a_v)) \in \{(2, k), (k, 2), (k-2, k), (k, k-2)\}$. The first case is when $L(a_u) = 2$ and $L(a_v) = k$. Take $L(b_j) = j+3$ for $j = 1, \dots, k-5$. A k - $L(2, 1)$ -labelling is obtained by setting $L(d_{l,j}) = L(d_{r,j}) = k$ and $L(c_{l,j}) = 0$ and $L(c_{r,j}) = 1$ for $1 \leq j \leq k-5$ and then using Lemma 12 to give a valid labelling of the rest of the graph.

A similar argument shows that we may take $(L(a_u), L(a_v)) = (k, 2)$.

The second case is $L(a_u) = k-2$ and $L(a_v) = k$. Take $L(b_j) = j+1$ for $j = 1, \dots, k-5$. A k - $L(2, 1)$ -labelling is obtained by setting $L(d_{l,j}) = 0$, $L(d_{r,j}) = k$ and $L(c_{l,j}) = k-1$ and $L(c_{r,j}) = 0$ for $1 \leq j \leq k-5$ and then using Lemma 12 to give a valid labelling of the rest of the graph.

A similar argument shows that we may take $(L(a_u), L(a_v)) = (k, k-2)$.

Next we show that there is no k - $L(2, 1)$ -labelling L of G_k with $L(u) = L(v) = 0$ and $(L(a_v), L(a_u)) \notin \{(2, k), (k, 2), (k-2, k), (k, k-2)\}$. Assume without loss of generality that $L(a_u) < L(a_v)$. Suppose first that $3 \leq L(a_u) \leq k-3$ and $L(a_v) = k$. Note that for any j by considering the proximity of b_j to u and a_v , we see that b_j cannot be labelled with $0, k-1$ or k . Furthermore we cannot have $b_j = 1$ because then $\{L(c_{l,j}), L(c_{r,j})\} = \{k, k-1\}$. But they are both at distance two from a_v which is labelled k , so this is not possible. So b_1, \dots, b_{k-5} must receive distinct labels from $\{2, \dots, k-2\} \setminus \{L(a_u) - 1, L(a_u), L(a_u) + 1\}$, but this only gives $k-6$ labels which is not enough.

Now suppose $L(a_v) \neq k$ and $2 \leq L(a_u) \leq L(a_v) - 2 \leq k-3$. Note that for any j , b_j cannot be labelled with 0 . Furthermore b_j cannot be labelled k as then $\{L(c_{l,j}), L(c_{r,j})\} = \{0, 1\}$ and $L(d_{l,j}) = L(d_{r,j}) = k$ but this is invalid. So b_1, \dots, b_{k-5} must receive distinct labels from $\{1, \dots, k-1\} \setminus \{L(a_u) - 1, L(a_u), L(a_u) + 1, L(a_v) - 1, L(a_v), L(a_v) + 1\}$. This is only possible if $L(a_u) = k-3$ and $L(a_v) = k-1$. Then $\{L(b_1), \dots, L(b_{k-5})\} = \{1, \dots, k-5\}$. However if $L(b_j) = 1$ then $\{L(c_{l,j}), L(c_{r,j})\} = \{k-1, k\}$. But this is invalid as $L(a_v) = k-1$. \square

Analogously we obtain the following lemma.

Lemma 14. *There exists a k - $L(2, 1)$ -labelling L of G_k with $L(u) = L(v) = k$ if and only if $(L(a_v), L(a_u)) \in \{(2, 0), (0, 2), (k-2, 0), (0, k-2)\}$.*

Lemma 15. *There exists a k - $L(2, 1)$ -labelling L of G_k with $L(u) = k$ and $L(v) = 0$ if and only if $(L(a_v), L(a_u)) = (k-1, 1)$.*

Proof. Let $l_1 = \min\{L(a_u), L(a_v)\}$ and $l_2 = \max\{L(a_u), L(a_v)\}$. The vertices b_1, \dots, b_{k-5} must be labelled with distinct labels from $S = \{1, \dots, k-1\} \setminus \{l_1-1, l_1, l_1+1, l_2-1, l_2, l_2+$

1}. The only way that this set can contain $k - 5$ elements is if $(l_1, l_2) = (1, k - 1)$, $(l_1, l_2) = (1, 3)$ or $(l_1, l_2) = (k - 3, k - 1)$.

We first show that there exists a k - $L(2, 1)$ -labelling L of G_k with $L(u) = k, L(v) = 0$ and $(L(a_v), L(a_u)) = (k - 1, 1)$.

We need $\{L(b_1), \dots, L(b_{k-5})\} = \{3, \dots, k - 3\}$. Then let $L(c_{l,j}) = 0, L(d_{l,j}) = k, L(c_{r,j}) = k$ and $L(d_{l,j}) = 0$ for $1 \leq j \leq k - 5$. So by Lemma 12 we obtain a valid labelling.

Next we show that there is no k - $L(2, 1)$ -labelling of G_k with $L(u) = k$ and $L(v) = 0$ and $(L(a_v), L(a_u)) \neq (k - 1, 1)$.

Assume that $l_1 = 1$ and $l_2 = 3$ so $L(a_v) = 3$ and $L(a_u) = 1$. Then the vertices b_1, \dots, b_{k-5} must take distinct labels from $\{5, \dots, k - 1\}$. However if b_j is labelled $k - 1$ the only label $c_{l,j}$ and $c_{r,j}$ can be labelled with is 0 but $c_{l,j}$ and $c_{r,j}$ must have distinct labels. Therefore this labelling is not possible.

Now suppose $l_1 = k - 3$ and $l_2 = k - 1$ then $L(a_v) = k - 1$ and $L(a_u) = k - 3$. Then the vertices b_1, \dots, b_{k-5} must take distinct labels from $\{1, \dots, k - 5\}$. However if b_j is labelled 1 the only label $c_{l,j}$ and $c_{r,j}$ can be labelled with is k but $c_{l,j}$ and $c_{r,j}$ must have distinct labels. Therefore this labelling is not possible. \square

3.4 Summary

The following theorem summarises the results of this section and follows immediately from Lemmas 10, 11, 13, 14 and 15.

Theorem 16. *Let $k \geq 4$ be fixed. There is a k - $L(2, 1)$ -labelling L of G_k with $L(u), L(v) \in \{0, k\}$ if and only if the following conditions are satisfied.*

1. *If $(L(u), L(v)) = (0, 0)$ then $(L(a_u), L(a_v)) \in \{(2, k), (k, 2), (k - 2, k), (k, k - 2)\}$.*
2. *If $(L(u), L(v)) = (k, k)$ then $(L(a_u), L(a_v)) \in \{(2, 0), (0, 2), (k - 2, 0), (0, k - 2)\}$.*
3. *If $(L(u), L(v)) = (k, 0)$ then $(L(a_u), L(a_v)) = (1, k - 1)$.*
4. *If $(L(u), L(v)) = (0, k)$ then $(L(a_u), L(a_v)) = (k - 1, 1)$.*

4 Planar k - $L(2, 1)$ -labelling is NP-complete for $k \geq 4$

We reduce Planar Cubic Two-Colourable Perfect Matching to Planar k - $L(2, 1)$ -Labelling. Suppose we are given a cubic planar graph G corresponding to an instance of Planar Cubic Two-Colourable Perfect Matching. From G we construct a graph K which has the property that K has a k - $L(2, 1)$ -labelling if and only if G has a two-coloured perfect matching.

In order to show this we also construct an auxiliary graph H and define what we call a *coloured orientation*. Then we show that G has a two-coloured perfect matching if and only if H has a coloured orientation and finally that H has a coloured orientation if and only if K has a k - $L(2, 1)$ -labelling.

H is obtained by replacing every edge of G with the gadget as depicted in Fig. 10, where the endpoints of the edge being replaced are u, v .

The gadget has two special vertices labelled *in* and *out*, which we call the *invertex* and the *outvertex* and we explain in a moment. The edges incident with them are called the *inedge* and *outedge*, respectively. We use the phrase *auxiliary edge* to refer to a subgraph of H that has replaced an edge of G , that is, any of the copies of the gadget from Fig. 10. A coloured orientation of an auxiliary graph H is a colouring of the vertices of H with black and white and an orientation of some of the edges satisfying certain properties. The indegree and outdegree of a vertex v are the number of edges oriented towards v and the number of edges oriented away from v , respectively. Unoriented edges are not counted towards indegree and outdegree. A coloured orientation must satisfy the following properties.

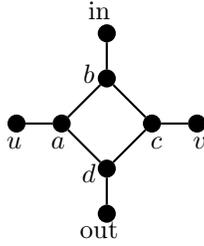


Figure 10: An auxiliary edge.

- Every vertex is adjacent to at most one vertex of the opposite colour.
- An edge is oriented if and only if it is monochromatic.
- Every vertex except those labelled *out* has outdegree at most two and indegree at most one.
- Every vertex labelled *out* has indegree zero.

We say a coloured orientation is *good* if every vertex of degree three is adjacent to precisely one vertex of the opposite colour and has indegree and outdegree one.

Lemma 17. *Let G be a cubic planar graph and let H be the corresponding auxiliary graph. If G has a two-coloured perfect matching then H has a good coloured orientation.*

Proof. First colour the vertices of H that were present in G with the same colour that they receive in G .

We next colour the vertices of auxiliary edges where both endpoints of the corresponding edge in G receive the same colour. The in- and outvertex receive the same colour as the endpoints of the corresponding edge in G and the vertices on the four-cycle receive the opposite colour. We orient this cycle to form a directed cycle, see Fig. 11.

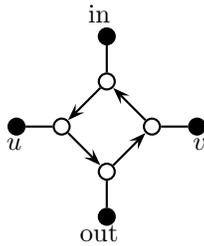


Figure 11: Good coloured orientation of an auxiliary edge if u and v receive the same colour.

Now we colour all the other vertices and orient edges as follows. Vertices remaining uncoloured all belong to auxiliary edges for which the endpoints of the corresponding edge in G are coloured differently in G . We start by choosing an auxiliary edge e between a black vertex v and a white vertex w_1 of G . In G , v has two white neighbours and one black neighbour. Call the other white neighbour w_2 . Colour the outvertex of e black and orient the edge incident with it away from the outvertex. Now follow the shortest path from the outvertex, through v and to the invertex of the auxiliary edge vw_2 . Colour every uncoloured vertex on this path black and orient every edge consistently with the path. At each stage of this colouring/orientation process we will colour and orient a path like this from an outvertex of an auxiliary edge, through a vertex present in G to an invertex of a neighbouring auxiliary edge, see Fig. 12. It

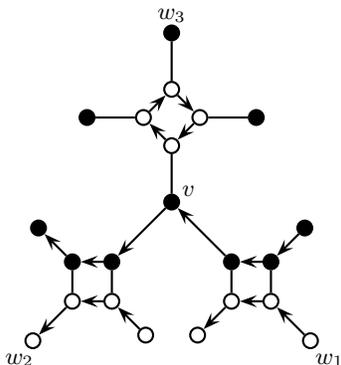


Figure 12: Assignment of a good coloured orientation on H .

only remains to describe how to choose the outvertex and invertex pair forming the endpoints of each path. The first pair is chosen as above. Otherwise, if at some stage, we colour an invertex of an auxiliary edge f with colour c and the outvertex f is still uncoloured then at the next stage we form a path from the outvertex of f colouring the vertices on it with the opposite colour to c . If the outvertex of f has already been coloured then we choose another auxiliary edge for which the outvertex is uncoloured. This method ensures that at each stage there is at most one auxiliary edge with the outvertex coloured and the invertex uncoloured and at most one auxiliary edge with the outvertex not coloured but the invertex coloured. Such an uncoloured outvertex is always the next one to be coloured.

Due to the construction process, every vertex of H which is also present in G is adjacent to exactly one vertex of the opposite colour and has indegree and outdegree one. Clearly the same is true for all vertices of degree three of auxiliary edges where both endpoints of the corresponding edge in G receive the same colour. Now consider an auxiliary edge which corresponds to a dichromatic edge e in G . Due to the colouring/orientation process the invertex and outvertex must receive opposite colours and the shortest path from each of them to the endpoints of e with the same colour is monochromatic. It follows that all vertices of degree three on the auxiliary edge must be adjacent to exactly one vertex of the opposite colour and have indegree and outdegree one, see Fig. 12. Therefore the method yields a good coloured orientation. \square

Lemma 18. *Let G be a cubic planar graph and let H be the corresponding auxiliary graph. If H has a coloured orientation, then it has a good coloured orientation.*

Proof. Consider the possible coloured orientations of an auxiliary edge. Because each vertex is adjacent to at most one of the opposite colour the only ways in which the four-cycle of an auxiliary edge may be coloured are with all four vertices receiving the same colour or with a pair of adjacent vertices receiving one colour and the other pair receiving the opposite colour. In the first case we may change the colours of the invertex and outvertex (if necessary) to be the opposite colour to that of the vertices in the four-cycle. (We also remove the orientation of the inedge and outedge if necessary.) In this way both the inedge and the outedge are dichromatic. In the second case the fact that each vertex is adjacent to at most one vertex of the opposite colour forces both the invertex and the outvertex to have the same colour as their neighbour.

From now on we will assume we have a coloured orientation with each auxiliary edge being coloured in this way. We will show that if H has a coloured orientation then it has a good coloured orientation. Let H' be formed from H by deleting all the

dichromatic, or equivalently unoriented edges and consider a connected component C of H' . In H' every vertex has outdegree at most two and indegree at most one. So C is either an isolated vertex, a directed circuit with a number of trees rooted on the circuit and directed away from the circuit or a directed rooted tree in which all edges are oriented away from the root. Bearing in mind the constraints on the in- and outdegree of the vertices, we see that every vertex of degree three in the auxiliary graph has total degree at least two in H' . Leaves of H' correspond to invertices and roots with degree one correspond to either invertices or outvertices. Notice that the isolated vertices of H' can only be invertices or outvertices and by the remarks at the beginning of the proof exactly half of the isolated vertices are invertices. Hence the numbers of invertices and outvertices appearing in H' that are not isolated are equal. So the number of leaves of H' is at most the number of roots of tree components. Consequently each connected component of H' , that is not just an isolated vertex, is either a path beginning at an outvertex and ending at an invertex or a directed circuit. So every vertex of degree three in the auxiliary graph has one out-neighbour, one in-neighbour and one incident unoriented edge. Therefore the coloured orientation is good. \square

Lemma 19. *Let G be a cubic planar graph and let H be the corresponding auxiliary graph. If H has a good coloured orientation then G has a two-coloured perfect matching.*

Proof. Consider a vertex v of G and let w_1, w_2 and w_3 be its neighbours in G . Suppose without loss of generality that v is coloured black in the good coloured orientation of H . We will show that in H , two of the vertices w_1, w_2, w_3 are coloured white and one is coloured black. Then we only need to assign to any vertex in G the colour it receives in the good coloured orientation in H to obtain a two-coloured perfect matching of G .

Vertex v has two black neighbours and one white neighbour in H . In the proof of Lemma 18 we showed that in a good coloured orientation a terminal vertex of an auxiliary edge receives the opposite colour to the unique neighbour in the auxiliary edge of the other terminal vertex of the auxiliary edge. Thus two of the vertices w_1, w_2, w_3 must be coloured white and the other one black. \square

Now given an instance G of Planar Cubic Two-Colourable Perfect Matching, we define an instance K of k - $L(2, 1)$ -labelling. First form the auxiliary graph H . For every vertex v of H add sufficient vertices of degree one with edges joining them to v to ensure that v has degree $k - 1$. Now replace each edge that was originally present in H by the gadget G_k identifying the vertices u, v of G_k with the two endpoints of edges of H being replaced. Finally for each outvertex v choose a neighbour w of v with degree one and add $k - 2$ vertices of degree one joined to w . To illustrate this, suppose that in G , v is adjacent to w_1, w_2, w_3 . In Fig. 13 we show how the neighbourhood of v is modified in K . Note that K can be constructed from G in time $O(n)$.

Lemma 20. *Let G be a cubic planar graph and let H be the corresponding auxiliary graph. Let K be the instance of k - $L(2, 1)$ -labelling constructed from G as described above. Then H has a good coloured orientation if and only if K has a k - $L(2, 1)$ -labelling.*

Proof. Suppose that K has a k - $L(2, 1)$ -labelling L . We now describe how to obtain a coloured orientation of H from L . Any vertex in H corresponds to a vertex of degree $k - 1$ in K and so must be coloured 0 or k . Colour a vertex of H white if it corresponds to a vertex labelled 0 in K and black if it corresponds to vertex labelled k in K .

We orient some of the edges of H as follows. If uv is an edge of H then there is a path u, a_u, a_v, v between u, v in K where u is adjacent to a_u and v is adjacent to a_v . Orient the edge uv from u to v if and only if $a_v \in \{0, k\}$ and orient it from v to u if and only if $a_u \in \{0, k\}$. From Theorem 16 it follows that in our colouring of H , each vertex of H is adjacent to at most one vertex of the opposite colour, and an edge is oriented if and only if it joins two vertices of the same colour. Consider a vertex $v \in H$. All neighbours of v in K must receive different colours, so in H , v has at most one incoming

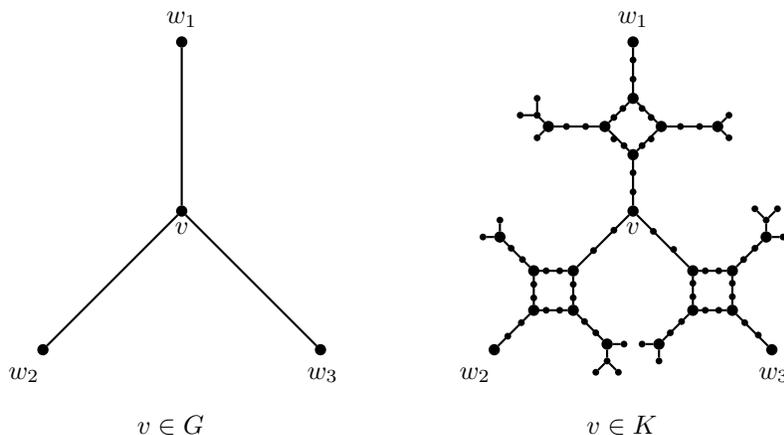


Figure 13: Construction of graph K from G for $k = 4$.

edge, and at most two outgoing edges. Finally let u be an outvertex of H . Then u is part of exactly one copy of the gadget G_k and has a neighbour w of degree $k - 1$ that is not part of this copy of G_k . We have $\{L(u), L(w)\} = \{0, k\}$ which means that no other neighbour of u is labelled 0 or k and hence u has indegree 0. Therefore H has a coloured orientation and by Lemma 18, H has a good coloured orientation.

Now suppose that H has a good coloured orientation. We will show how to construct a k - $L(2, 1)$ -labelling L of K . First label all vertices v in K that appear in H , so that $L(v) = 0$ if v is coloured white in H and otherwise $L(v) = k$. Next give labels to all the remaining vertices that appear in a copy of the gadget G_k . Let uv be an edge of H and suppose without loss of generality that $L(u) = 0$. Let u, a_u, a_v, v be the path of length three from u to v in K . If uv is not oriented, let $L(a_u) = k - 1$ and $L(a_v) = 1$. If uv is oriented from u to v then let $L(a_u) = 2, L(a_v) = k$ and if uv is oriented from v to u then let $L(a_u) = k, L(a_v) = 2$. Furthermore if $w \in V(H)$ then, because we start from a good coloured orientation of H , its three neighbours in K receive different labels. Then Theorem 16 shows that L may be extended so that any vertex appearing in a copy of G_k receives a label. For each outvertex, its neighbour of degree $k - 1$ must be labelled. This can be done because each edge in H adjacent to an outvertex x is oriented away from x , so one of the labels 0, k is always available. Finally the vertices of degree one form an independent set and are all adjacent to vertices of degree $k - 1$ that have received label 0 or k . So they may be labelled. Hence K has a k - $L(2, 1)$ -labelling. \square

We now return to the main problem of this paper. The following theorem which is the main statement of this paper follows immediately from Theorem 9 and Lemmas 17, 18, 19 and 20.

Theorem 21. *Problem 1 is NP-complete.*

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