Robust Stability of Two-Dimensional Uncertain Discrete Systems

Zidong Wang and Xiaohui Liu

***Abstract*—In this letter, we deal with the robust stability problem for linear two-dimensional (2-D) discrete time-invariant systems described by a 2-D local state–space (LSS) Fornasini–Marchesini second model. The class of systems under investigation involves parameter uncertainties that are assumed to be norm-bounded. We first focus on deriving the sufficient conditions under which the uncertain 2-D systems keep robustly asymptotically stable for all admissible parameter uncertainties. It is shown that the problem addressed can be recast to a convex optimization one characterized by linear matrix inequalities (LMIs), and therefore a numerically attractive LMI approach can be exploited to test the robust stability of the uncertain discrete-time 2-D systems. We further apply the obtained results to study the robust stability of perturbed 2-D digital filters with overflow nonlinearities.**

***Index Terms*—Linear matrix inequalities, overflow nonlinearities, parameter uncertainty, robust stability, two-dimensional discrete-time.**

I. Introduction

In the past decade, the stability analysis problem for two-dimensional (2-D) linear and nonlinear discrete systems (filters) has received considerable attention and various Lyapunov approaches have been proposed as effective tool. In particular, the constant 2-D Lyapunov equation has been investigated in [1] and [7] for the Roesser models. Moreover, the constant Lyapunov-type criterion has been given in [10] to guarantee the asymptotic stability of the 2-D linear Fornasini-Marchesini models and this result has been further improved in [13]. In [3] and [12], the Lyapunov stability problem has been investigated for 2-D digital filters with overflow nonlinearities. Also, the stabilization problem (or more advancely, the feedback control synthesis problem) of 2-D systems has been extensively studied and most relevant work have been concerned with solving linear polynomials or polynomial matrices in two variables (e.g., see [2], [8], and [11]).

On the other hand, since modeling error (implementation error) is very often the cause of the instability of the 2-D systems (2-D filters), the robust stability synthesis (or *stabilization*) issue has also begun to draw initial attention (e.g., see [6]). It is noticeable that, in one-dimensional (1-D) case, the important robust stability analysis problem for linear systems with parameter uncertainties has been a hot research topic over the last three decades. The robust filtering problem for the 1-D case has also been well studied (see [15]–[17] and the references therein). However, so far, there are very few results on the robust stability *analysis* problem for uncertain 2-D systems (filters), especially for uncertain 2-D systems (filters) with overflow nonlinearities.

In this letter, we first deal with the robust stability analysis problem for linear 2-D discrete time-invariant systems described by a 2-D local state-space (LSS) Fornasini–Marchesini second model. The class of systems under investigation involves parameter uncertainties that are assumed to be norm-bounded. We focus on deriving the sufficient conditions under which the uncertain 2-D systems keep robustly asymptotically stable for all admissible parameter uncertainties. It is shown that the problem addressed can be recast to a convex optimization one characterized by linear matrix inequalities (LMIs), and therefore a numerically attractive LMI approach [4], [9] can be employed to test the robust stability of the uncertain linear discrete-time 2-D systems. Furthermore, by using the similar method, we tackle the robust stability analysis problem for perturbed 2-D digital filters with overflow nonlinearities. It should be pointed out that, in the past few years, linear matrix inequalities (LMIs) have gained much attention for their computational tractability and usefulness in control engineering and the number of control problems that can be formulated as LMI problems is large and continue to grow [4]. The LMIs can now be solved efficiently by the powerful Matlab LMI Toolbox [9].

II. Problem Formulation

Consider the uncertain 2-D discrete system described by the 2-D LSS model [14]

$$x(i+1,j+1) = (A_1+\Delta A_1)\,x(i,j+1) + (A_2+\Delta A_2)\,x(i+1,j) \quad (1)$$

where $x(i,j) \in \mathbb{R}^n$ is the state vector and $A_1,\ A_2 \in \mathbb{R}^{n\times n}$ are known constant matrices.

The matrices ΔA_1 and ΔA_1 represent parametric perturbations in the system state matrices and are assumed to be of the following form (e.g., see [6])

$$[\,\Delta A_1 \quad \Delta A_2\,] = MF(i,j)\,[\,N_1 \quad N_2\,] \quad (2)$$

where $F(i,j) \in \mathbb{R}^{i\times j}$ is an uncertain matrix with Lebesgue measurable elements bounded by

$$F^T(i,j)F(i,j) \le I \quad (3)$$

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The authors are with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex UB8 3PH, U.K. (e-mail: Zidong.Wang@brunel.ac.uk).

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and M, N_1, and N_2 are known constant matrices of appropriate dimensions that specify how the elements of the nominal matrices A_1 and A_2 are affected by the uncertain parameters in $F(i,j)$. ΔA_1 and ΔA_2 are said to be admissible if both (2) and (3) hold.

The system (1) is said to be robustly asymptotically stable if it is asymptotically stable [with respect to the equilibrium (0,0)] for all admissible uncertainties described by (2) and (3). The objective of this letter is to determine whether the 2-D discrete system (1) is robustly asymptotically stable, i.e., to derive the easy-to-test conditions under which the system (1) remains robustly asymptotically stable.

III. Main Results and Proofs

To begin with, we recall several important lemmas that will be frequently used in the derivation of our main results. The first one is well known on the asymptotic stability of the 2-D nominal discrete systems.

Lemma 1: [10]: The nominal 2-D discrete LSS system

$$x(i+1,j+1) = A_1 x(i,j+1) + A_2 x(i+1,j) \tag{4}$$

is asymptotically stable if there exist an $n \times n$ symmetric matrix $H > 0$, positive scalars α and β satisfying $\alpha + \beta = 1$ such that

$$A^T H A - Q < 0 \tag{5}$$

where

$$A = [\, A_1 \quad A_2 \,], \quad Q = \begin{bmatrix} \alpha H & 0 \\ 0 & \beta H \end{bmatrix}. \tag{6}$$

The next lemma is the so-called Schur Complement Lemma.

Lemma 2: Given constant matrices M, L and R of appropriate dimensions where M and R are symmetric and $R > 0$, then $M + L^T R L < 0$ if and only if

$$\begin{bmatrix} M & L^T \\ L & -R^{-1} \end{bmatrix} < 0 \text{ or } \begin{bmatrix} -R^{-1} & L \\ L^T & M \end{bmatrix} < 0.$$

The following lemma reveals that the robust stability of a "discrete-time system" subjected to the norm-bounded uncertainty can be guaranteed by the existence of the positive definite solution to a matrix inequality.

Lemma 3: [18]: Let $A \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times i}$, $N \in \mathbb{R}^{j\times n}$, $Q = Q^T \in \mathbb{R}^{n\times n}$ be given matrices and $F \in \mathbb{R}^{i\times j}$ be any matrix satisfying $F^T F \le I$. Then there exists a positive definite matrix $P > 0$ meeting

$$(A + MFN)^T P (A + MFN) + Q < 0 \tag{7}$$

if and only if there exist a scalar $\varepsilon > 0$ and a positive definite matrix $P > 0$ such that

$$\begin{bmatrix} -P^{-1} + \varepsilon M M^T & A \\ A^T & \varepsilon^{-1} N^T N + Q \end{bmatrix} < 0. \tag{8}$$

For the sake of simplicity, we make the following definitions:

$$N := [\, N_1 \quad N_2 \,], \quad \Delta A := [\, \Delta A_1 \quad \Delta A_2 \,] \tag{9}$$

$$\Omega := \begin{bmatrix} M & 0 \\ 0 & N^T \end{bmatrix}, \quad \Gamma := \begin{bmatrix} \varepsilon I & 0 \\ 0 & \varepsilon^{-1} I, \end{bmatrix} \tag{10}$$

$$\Theta := -\alpha H + \sigma N_1^T N_1, \quad \Sigma := -\beta H + \sigma N_2^T N_2 \tag{11}$$

Our first result given below shows that the uncertain discrete 2-D system (1) is robustly asymptotically stable if there exists a positive definite solution to a matrix inequality involving a scalar parameter ε.

Theorem 1: Consider the linear uncertain 2-D discrete-time LSS system (1). Assume that the uncertainties ΔA_1 and ΔA_2 satisfy (2)–(3). Then the system (1) is robustly asymptotically stable if, for some positive scalars α and β satisfying $\alpha + \beta = 1$, there exists a positive scalar ε such that the matrix inequality

$$\begin{bmatrix} -H & HA_1 & HA_2 & HM & 0 \\ A_1^T H & -\alpha H & 0 & 0 & N_1^T \\ A_2^T H & 0 & -\beta H & 0 & N_2^T \\ M^T H^T & 0 & 0 & -\varepsilon^{-1} I & 0 \\ 0 & N_1 & N_2 & 0 & -\varepsilon I \end{bmatrix} < 0 \tag{12}$$

has a positive definite solution $H \in \mathbb{R}^{n\times n}$.

Proof: It follows from (2) that

$$[\, A_1 + \Delta A_1 \quad A_2 + \Delta A_2 \,] = A + \Delta A = A + MFN. \tag{13}$$

Then, it is easy to see from Lemma 1 that the uncertain discrete 2-D system (1) is robustly asymptotically stable if, for some positive scalars α and β satisfying $\alpha + \beta = 1$, there exists a positive definite matrix $H \in \mathbb{R}^{n\times n}$ meeting

$$(A + MFN)^T H (A + MFN) - Q < 0. \tag{14}$$

Furthermore, it results from Lemma 3 that, the inequality (14) holds if and only if, for some positive scalars α and β satisfying $\alpha + \beta = 1$, there exists a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} -H^{-1} + \varepsilon M M^T & A \\ A^T & \varepsilon^{-1} N^T N - Q \end{bmatrix} < 0 \tag{15}$$

has a positive definite solution $H \in \mathbb{R}^{n\times n}$.

Note that the inequality (15) can be rewritten as

$$\begin{bmatrix} -H^{-1} & A \\ A^T & -Q \end{bmatrix} + \Omega \Gamma \Omega^T < 0 \tag{16}$$

where Ω and Γ are defined in (10).

It then follows from the Schur Complement Lemma (Lemma 2) that the inequality (16) holds if and only if

$$\begin{bmatrix} -H^{-1} & A & M & 0 \\ A^T & -Q & 0 & N^T \\ M^T & 0 & -\varepsilon^{-1} I & 0 \\ 0 & N & 0 & -\varepsilon I \end{bmatrix} < 0. \tag{17}$$

Pre- and postmultiplying the inequality (17) by $\mathrm{diag}\{H, I, I, I\}$, respectively, yield

$$\begin{bmatrix} -H & HA & HM & 0 \\ A^T H & -Q & 0 & N^T \\ M^T H & 0 & -\varepsilon^{-1} I & 0 \\ 0 & N & 0 & -\varepsilon I \end{bmatrix} < 0 \tag{18}$$

which, by means of the definitions (9) and (10), can be expressed as the following:

$$\begin{bmatrix} -H & HA_1 & HA_2 & HM & 0 \\ A_1^T H & -\alpha H & 0 & 0 & N_1^T \\ A_2^T H & 0 & -\beta H & 0 & N_2^T \\ M^T H^T & 0 & 0 & -\varepsilon^{-1} I & 0 \\ 0 & N_1 & N_2 & 0 & -\varepsilon I \end{bmatrix} < 0 \tag{19}$$

i.e., (12) holds. This completes the proof of our first result. ■

Remark 1: It should be pointed out that the characterization condition (12) cannot easily be used, since it involves both ε^{-1} and ε and is therefore not a linear matrix inequality. This brings considerable difficulties in practical application because the powerful Matlab LMI toolbox [4], [9] cannot be utilized. This awkward situation motivates us to further convert (12) into an LMI. On the other hand, we may also notice that the matrix inequality (12) is not jointly linear on α ($\beta = 1 - \alpha$) and H. Fortunately, this minor "problem" can be ignored, as $\alpha \in (0,1)$, and we can solve (12) by increasing $\alpha > 0$ with an acceptable step size.

The following theorem provides the sufficient conditions for the robust asymptotic stability of the uncertain 2-D discrete system (1) in terms of an LMI.

Theorem 2: The 2-D discrete-time LSS system (1) with parameter uncertainties satisfying (2) and (3) is robustly asymptotically stable if, for some positive scalars α and β ($\alpha + \beta = 1$), there exist a positive scalar σ and a positive definite matrix $H > 0$ such that the following linear matrix inequality

$$\Upsilon := \begin{bmatrix} -H & HA_1 & HA_2 & HM \\ A_1^T H & \Theta & \sigma N_1^T N_2 & 0 \\ A_2^T H & \sigma N_2^T N_1 & \Sigma & 0 \\ M^T H^T & 0 & 0 & -\sigma I \end{bmatrix} < 0 \quad (20)$$

holds, where Θ and Σ are defined in (11).

Proof: First, for technical convenience, we set $\sigma := \varepsilon^{-1}$. Then, the Schur Complement Lemma (Lemma 2) implies that (12) is true if and only if the following inequality

$$\begin{bmatrix} -H & HA_1 & HA_2 & HM \\ A_1^T H & -\alpha H & 0 & 0 \\ A_2^T H & 0 & -\beta H & 0 \\ M^T H^T & 0 & 0 & -\varepsilon^{-1} I \end{bmatrix} + \begin{bmatrix} 0 \\ N_1^T \\ N_2^T \\ 0 \end{bmatrix} \cdot \varepsilon^{-1} I \cdot [0 \quad N_1 \quad N_2 \quad 0] = \Upsilon < 0 \quad (21)$$

holds for some positive scalar σ ($\sigma = \varepsilon^{-1}$) and positive definite matrix H, i.e., (20) holds. This proves the theorem. ■

Remark 2: It is clear that, for a given scalar $\alpha > 0$, the inequality (20) is jointly linear on both H and σ and therefore the Matlab LMI Toolbox can be exploited to investigate the robust asymptotic stability of the discrete-time 2-D system (1).

The algorithm for testing the robust stability can be described as follows.

- Give an initial value $\alpha > 0$, which can be small enough.
- Solve the LMI (20) for $\sigma > 0$ and $H > 0$ by using the LMI Toolbox. If there are a positive scalar σ and a positive definite matrix H solving (20), then the uncertain 2-D system (1) is robustly asymptotically stable, and the procedure stops; if there are no solutions, then increase $\alpha > 0$ by an acceptable step size, and solve LMI (20) again.
- If α approaches one eventually, then it means that the uncertain discrete-time 2-D system is *not* robustly asymptotically stable.

Remark 3: It is worth mentioning that although the uncertainty in this letter is assumed to be norm-bounded, it is not difficult to consider more general description of the uncertainty within the same framework developed in this letter. For example, it has been shown in [5] that the more involved polytopic convex uncertainty can be taken into account by introducing some LMIs and the corresponding robustness can be guaranteed when the solutions to certain LMIs are known to exist. Thus, following along the line of the proof of Theorem 2, we will be able to further consider the robust stability analysis problem for more general uncertain 2-D systems in terms of LMIs and the present techniques can all be applied.

IV. Stability Analysis of 2-D Digital Filters With Overflow Nonlinearities

This section aims at showing the applicability of the results we obtained in the previous section. We shall choose the 2-D digital filters with overflow nonlinearities as an example.

Consider the zero-input 2-D nonlinear digital filters with implementation error as follows:

$$x(i+1, j+1) = f\left[(A_1 + \Delta A_1)\, x(i, j+1) + (A_2 + \Delta A_2)\, x(i+1, j)\right] \quad (22)$$

where $x(i,j) \in \mathbb{R}^n$ is the state vector; $A_1,\ A_2 \in \mathbb{R}^{n\times n}$ are known constant matrices; and the perturbation matrices ΔA_1 and ΔA_1 (introduced as implementation error) have the same structure as in (2) and (3). The nonlinear $f : \mathbb{R}^n \to \mathbb{R}^n$ representing overflow effects in 2-D digital filters (22) is defined by $f(x) = [\phi(x_1), \ldots, \phi(x_N)]^T$ where $\phi : \mathbb{R} \to [-1, 1]$ is piecewise continuous and is defined by [12]

$$\begin{cases} L \le \phi(x_i) \le L_1, & x_i > 1 \\ \phi(x_i) = x_i, & -1 \le x_i \le 1 \\ -L_2 \le \phi(x_i) \le -L, & x_i < -1 \end{cases} \quad (23)$$

where $-1 \le L \le L_1, L_2 \le 1$.

Remark 4: In practice, finite word-length realizations of digital filters often result in systems which are inherently nonlinear, and the asymptotic stability of such filters is of significance. As shown in [12], the class of overflow nonlinearities described in (23) constitute a generalization of the usual types of overflow arithmetic employed in applications such as zeroing, two's complement, triangular and saturation overflow characteristics.

Remark 5: Note that when implementing the digital filters, it is usually unavoidable to bring some implementation error, and sometimes this will lead to the instability of the digital filters, since the stability of the digital filters is very sensitive to the filter parameters. Thus, we introduce the terms ΔA_1 and ΔA_2 to account for the implementation error. When the perturbations disappear (i.e., $\Delta A_1 = 0$ and $\Delta A_2 = 0$), the system (22) will recover the one studied in [12].

Our goal in this section is to establish the conditions for the robust Lyapunov stability of the perturbed nonlinear 2-D filter (22). To do this, we first introduce the following result which offers the stability conditions for the *nominal* 2-D filter (22).

Lemma 4: [12]: Consider the nominal zero-input 2-D nonlinear digital filter (22) (setting $\Delta A_1 = 0$ and $\Delta A_2 = 0$). If there exists a positive definite matrix $H \in \mathbb{R}^{n\times n}$ such that $f^T(x) H f(x) < x^T H x$ (for all $x \in \mathbb{R}^n$, $x \notin D^n := \{x \in \mathbb{R}^n : -1 \le x_i \le 1\}$) and

$$A^T H A - Q < 0 \quad (24)$$

where A and Q are defined in (6) and $\alpha + \beta = 1$ ($\alpha > 0$, $\beta > 0$), then the equilibrium $x_e = 0$ of the *nominal* 2-D digital filter (22) is globally asymptotically stable.

Now, let us consider the *perturbed* nonlinear 2-D filter (22). The following theorem establishes the sufficient conditions for the robust global asymptotic stability of the trivial solution of (22) in terms of an LMI. Hence, using the Matlab toolbox, it is straightforward to test whether the perturbed nonlinear 2-D system (22) remains globally asymptotically stable.

Theorem 3: The equilibrium $x_e = 0$ of the perturbed nonlinear 2-D filter (22) is globally asymptotically stable if, for some positive scalars α and β ($\alpha + \beta = 1$), there exist a positive scalar σ and a positive definite matrix $H = [h_{ij}]$ such that

$$(1+L)h_{ii} \geq 2 \sum_{j=1, j\neq i}^{n} \mid h_{ij} \mid, \qquad i = 1, \ldots, n \tag{25}$$

and

$$\begin{bmatrix} -H & HA_1 & HA_2 & HM \\ A_1^T H & \Theta & \sigma N_1^T N_2 & 0 \\ A_2^T H & \sigma N_2^T N_1 & \Sigma & 0 \\ M^T H^T & 0 & 0 & -\sigma I \end{bmatrix} < 0 \tag{26}$$

where Θ and Σ are defined in (11).

Proof: The proof is a combination of Lemma 1 and Corollary 2 of [12] and Theorem 2 in the previous section and is thus omitted. ■

Remark 6: In application, when we test the robust global asymptotic stability of the perturbed nonlinear 2-D filter (22), the procedure is similar to that given for the linear case. The only difference is that, in the nonlinear case, we should first solve the LMI (26) for a given α and also check if there is a solution $H > 0$ meeting (25). Therefore, the approach developed in this section is numerically practical.

V. Conclusion

This letter has studied the robust stability analysis problem for a class of uncertain discrete 2-D systems. We have first considered the linear case. Sufficient conditions have been derived to ensure the robust asymptotic stability of the uncertain 2-D systems, which are given in terms of the solutions to a linear matrix inequalities and are therefore easy to test. Furthermore, to show the application potentials of the results obtained, we have investigated the robust stability for a class of perturbed 2-D digital filters with overflow nonlinearities.

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