TR/3 July 1971

HIGH FREQUENCY DIFFRACTION BY A SOFT CIRCULAR DISC.

I THE PLANE WAVE AT NORMAL INCIDENCE

by

J. C. NEWBY

ABSTRACT

The far scattered field off the axis of symmetry of the disc is found for a high frequency, harmonic, normally incident, plane wave. The method used is due to Jones and involves the solution of a singular integral equation of the first kind for the field on the disc. This integral equation can be converted into an integral equation of the second kind which is of particular value at high frequencies. In the present work the known function in the equation is written in the form of a contour integral. A suitable change of unknown function then produces extensive cancellation and yields a single function fundamental to the problem. The detailed calculations of the far field give terms which are believed to be new. In executing these calculations some interesting relationships between the terms involved are demonstrated.

INTRODUCTION

Diffraction by a circular disc has received much attention in the past due mainly to the fact that it is the simplest diffraction problem with a finite diffracting edge, and can therefore be used to check theories applicable to more general shapes. The problem can be solved exactly in terms of spheroidal functions but the resulting series are difficult to use, particularly at high frequencies. More recent interest has been in high frequency work with most approaches being through integral equations, though Keller's Ray Theory [6,7,8,9] is based upon physical optics. Levine [10] and Levine and Wu [11] modified the kernel of the standard integral equation in order to use Wiener-Hopf techniques. Keller's Theory and the approach used by Levine and Wu are applicable to problems other than the circular disc but they both involve approximations.

An entirely new approach was made by Jones [1] and it is this which is used in the present work. The method involves the exact solution of a certain integral equation, the uniqueness of which has also been demonstrated. Jones found the high frequency scattering coefficient for a normally incident plane wave on both the soft [1] and hard [2] circular discs, and later considered the electromagnetic case [3]. In the present work the far field off the axis of symmetry is found in detail yielding terms which are believed to be new. Further terms can easily be calculated if required. The results are compared with terms calculated by Keller.

In dealing with the known term in the integral equation a different approach is used to that employed by Jones. This term is expressed as a contour integal. The form of the contour suggests a change of the unknown function which might simplify the analysis. A subsequent contour deformation produces extensive cancellation and yields a mare amenable known function containing a single term fundamental to the problem. The insight gained by this method leads to the solution of the obliquely incident plane wave problem, details of which are not included in the present work.

1. THE INTEGRAL EQUATION

We shall consider the case of a small amplitude harmonic sound wave falling upon a sound soft circular disc of radius a. Sound soft is taken to mean that the field on the disc vanishes.
The problem may be normalised by regarding the radius a as the unit of length. The disc is in the plane z = 0 with its centre at the origin of a cylindrical polar coordinate system (r, ø, z) and occupies the region $0 \leq r \leq 1$ after normalisation.
Assume the incident field u_0 to be independent of ø, and to be represented by u_0 (r,z) at the point (r,ø,z)with time dependence e^{iwt} being understood and omitted throughout. The total field $U(\underset{\sim}{R})$ at the point $\underset{\sim}{R}$ is the sum of the incident field $u_0(\underset{\sim}{R})$ at $\underset{\sim}{R}$ and the field $u_s(\underset{\sim}{R})$ at $\underset{\sim}{R}$ produced by scattering from the disc.

Both u and u_s are independent of ø since u_0 is independent of ø and the disc is placed symmetrically on the axis of the system. Then

$$u(\underset{\sim}{R}) = u_0(\underset{\sim}{R}) + u_s(\underset{\sim}{R}) \qquad (1)$$

and the field u satisfies the Helmholtz wave equation

$$(\nabla^2 + \alpha^2)u = 0.$$

The symabol $\alpha = ka$ is a non-dimensional number, being the product of wavenumber k and disc radius a. This product is large at high frequencies.
A direct application of Green's Theorem shows that $u_s(\underset{\sim}{R})$ is given by

$$u_s(\underset{\sim}{R}) = -\frac{1}{4\pi}\int_S f(r_1)\frac{e^{-i\alpha|\underset{\sim}{R}-\underset{\sim}{R}_1|}}{|\underset{\sim}{R}-\underset{\sim}{R}_1|}ds, \qquad (2)$$

where S is the unit circle, centre the origin, in the plane $z = 0$. The position vector $\underset{\sim}{R}_1$ is a point of S and

$$f(r_1) = \left[\frac{\partial u(r_1, z_1)}{\partial z_1}\right]_{z_1=0+} - \left[\frac{\partial r(r_1, z_1)}{\partial z_1}\right]_{z_1=0-} \qquad (3)$$

is the discontinuity in the normal derivative of the field across the disc.

If $\underset{\sim}{R}$ is also a point of S, and the boundary condition $u = 0$ on S is applied, the following integral equation for $f(r_1)$ is obtained,

$$u_0(r,0) = \frac{1}{4\pi}\int_S f(r_1)\frac{e^{-i\alpha\,|\underset{\sim}{R} - \underset{\sim}{R}_1|}}{|\underset{\sim}{R} - R_1|}\,ds, \qquad (4)$$

$$= \frac{1}{4\pi}\int_0^1 r_1 f(r_1)\int_0^{2\pi}\frac{e^{-i\alpha\alpha\{^2+r_1^2-2rr_1\cos(\phi-\phi_1)\}^{\frac{1}{2}}}}{\{r^2 + r_1^2 - 2rr_1\cos(\phi-\phi_1)\}^{\frac{1}{2}}}\,d\phi_1 dr_1.$$

Jones [1] has shewn that this equation may be converted into the form

$$\int_0^1 w\, f(w)\left\{\frac{e^{-i\alpha\alpha(\ -\nu)}}{w-\nu} + \frac{e^{-i\alpha\alpha(\ +\nu)}}{w+\nu}\right\}dw = G(\nu), \qquad (5)$$

Where

$$G(\upsilon) = 2(2\pi(2^{\frac{1}{2}}\frac{d}{d\nu}\int_0^{\nu}\frac{\cos\{\alpha\,(\upsilon^2 - x^2)^{\frac{1}{2}}\}}{(\upsilon^2 - x^2)^{\frac{1}{2}}}\,x \times$$

$$\times\frac{d}{dx}\int_0^x \frac{I_{-\frac{1}{2}}\{\alpha\,(x^2 - r^2)^{\frac{1}{2}}\}}{(x^2 - r^2)^{\frac{1}{4}}}\,r\,u_0(r,0)\,dr\,dx \qquad (6)$$

$I_{-\frac{1}{2}}$ is the modified Bessel function of order $-\frac{1}{2}$. By letting

$$\int_0^1 wf(w)\frac{e^{-i\alpha(w-\upsilon)}}{w-\upsilon}dw = F(\nu) \qquad (7)$$

Jones showed that the singular integral equation (5) for f(w) can be written as an integral equation of the second kind for F(v) of the form

$$F(\upsilon) = G(\upsilon) - \frac{1}{\pi}\left(\frac{\upsilon}{1+\upsilon}\right)^{\frac{1}{2}} e^{-i\alpha\upsilon}\int_0^1\left(\frac{1-t}{t}\right)^{\frac{1}{2}}\frac{F(t)}{t+v}e^{-i\alpha t}dt\ . \qquad (8)$$

Inversion of (7) and application of the condition that f(w) should be bounded at the origin shows that

$$wf(w) = \frac{1}{\pi^2}\left(\frac{w}{1-w}\right)^{\frac{1}{2}} e^{idw}\int_0^1\left(\frac{1-\upsilon}{\upsilon}\right)^{\frac{1}{2}}\frac{F(\upsilon)}{w-\upsilon}e^{-id\upsilon}d\upsilon\ . \qquad (9)$$

2. THE KNOWN FUNCTION

In order to continue the analysis the known function, G(v), and hence the incident field, $u_0(r,z)$, must be specified. For a normally incident plane wave from the negative z direction

$$u_0(r,\ z) = e^{-i\alpha z}, \tag{10}$$

and so

$$G(\upsilon) = 2(2\pi\alpha)^{\frac{1}{2}} \frac{d}{d\upsilon} \int_0^{u} \frac{\cos\{\alpha(\upsilon^2 - x^2)^{\frac{1}{2}}\}}{(\upsilon^2 - x^2)^{\frac{1}{2}}} x^{\frac{3}{2}} I_{-\frac{1}{2}}(\alpha x) dx \tag{11}$$

The function G(v) does not have a uniformly valid asymptotic expansion. Such an expansion would be necessary in order to find a satisfactory solution of the integral equation (8). Some features of G(v) will now be discussed which give an insight into the problem and suggest a course of investigation. Using certain identities for Bessel functions (e.g. Watson [12]) we may write in the usual notation

$$x^{3/2} I_{-\frac{1}{2}}(\alpha x) = e^{-\alpha x}\eta(\alpha x) - e^{\alpha x}\eta(-\alpha x), \tag{12}$$

where

$$\eta(\alpha x) = \frac{1}{\pi} x^{3/2} e^{\alpha x} K_{-\frac{1}{2}}(\alpha x) = \frac{x}{(2\pi\alpha)^{\frac{1}{2}}} \tag{13}$$

Using this notation and the exponential form of the cosine G(v) can be written

$$G(v) = (2\pi\alpha)^{\frac{1}{2}} \frac{d}{d\upsilon} \left\{ \int_0^{\upsilon} \frac{e^{i\alpha\{(\upsilon^2 - x^2)^{\frac{1}{2}} + ix\}}}{(\upsilon^2 - x^2)^{\frac{1}{2}}} \eta(\alpha\alpha x) d + \right.$$

$$+ \int_0^{\upsilon} \frac{e^{-i\alpha\{(\upsilon^2 - x^2)^{\frac{1}{2}} - ix\}}}{(\upsilon^2 - x^2)^{\frac{1}{2}}} \eta(\alpha x) dx - \int_0^{\upsilon} \frac{e^{i\alpha\{(\upsilon^2 - x^2)^{\frac{1}{2}} - ix\}}}{(\upsilon^2 - x^2)^{\frac{1}{2}}} \eta(-\alpha x) dx -$$

$$\left. - \int_0^{\upsilon} \frac{e^{-i\alpha\{(\upsilon^2 - x^2)^{\frac{1}{2}} + ix\}}}{(\upsilon^2 - x^2)^{\frac{1}{2}}} \eta(-\alpha x) dx \right\},$$

In the first integral the substitution

$$(v^2 - x^2)^{\frac{1}{2}} + ix = -y$$

is made in order to simplify the exponential.
This integral may then be written

$$\int_0^{\upsilon} \frac{e^{i\alpha\{(\upsilon^2 - x^2)^{\frac{1}{2}} + ix\}}}{(\upsilon^2 - x^2)^{\frac{1}{2}}} \eta(\alpha x)dx = -i\int_{-\upsilon}^{-i\upsilon} \frac{ef^{-i\alpha y}}{y} \eta\left\{\frac{i\alpha}{2y}(y^2 - \upsilon^2)\right\}dy.$$

Similar changes in each of the other integrals are made so that all or the exponentials are reduced to the form $e^{-i\alpha\, y}$.
All the integrands are then the same, and

$$G(\upsilon) = (2\pi 2\pi^{\frac{1}{2}} \frac{d}{d\nu}\left\{\left(-i\int_{-\nu}^{-i\nu} + i\int_{\nu}^{-i\nu} - i\int_{-\nu}^{i\nu} + i\int_{\nu}^{i\nu}\right)\frac{e^{-i\alpha y}}{y}\eta\{\frac{i\alpha}{2y}(y^2 - \upsilon^2)\}dy\right\}, \qquad (14)$$

The contour in the y-plane is illustrated in figure 1.
There are two paths of integration from -v to +v, one being above the pole at the origin, and the other below it. The form of the contour and the exponential nature of the integrand suggest extending the contour from v to -i$^{\infty}$. We therefore define a new function $\mathscr{G}(\upsilon)$ by

$$\mathscr{G}(\upsilon) = -2i(2\pi\alpha)^{\frac{1}{2}} \frac{d}{d\nu}\int_{\upsilon}^{-i\infty} \frac{e^{-i\alpha y}}{y}\eta\left\{\frac{i\alpha}{2y}(y^2 - \upsilon^2)\right\}dy \qquad (15)$$

$$= -2(2\Pi\alpha)^{\frac{1}{2}} \frac{d}{d\nu}\int_0^{\infty} \frac{e^{-i\alpha}(\nu^2 - x^2)^{\frac{1}{2}}}{(\nu^2 - x^2)^{\frac{1}{2}}} \frac{1}{\pi} x^{\frac{3}{2}} K_{-\frac{1}{2}}(\alpha x)\, dx$$

in the original coordinates. The function $\mathscr{G}(\upsilon)$ is defined in such a way that when added to G(v) it completes the contours above and below the pole to -i∞. This enables a contour deformation to take place as illustrated in figure 2.

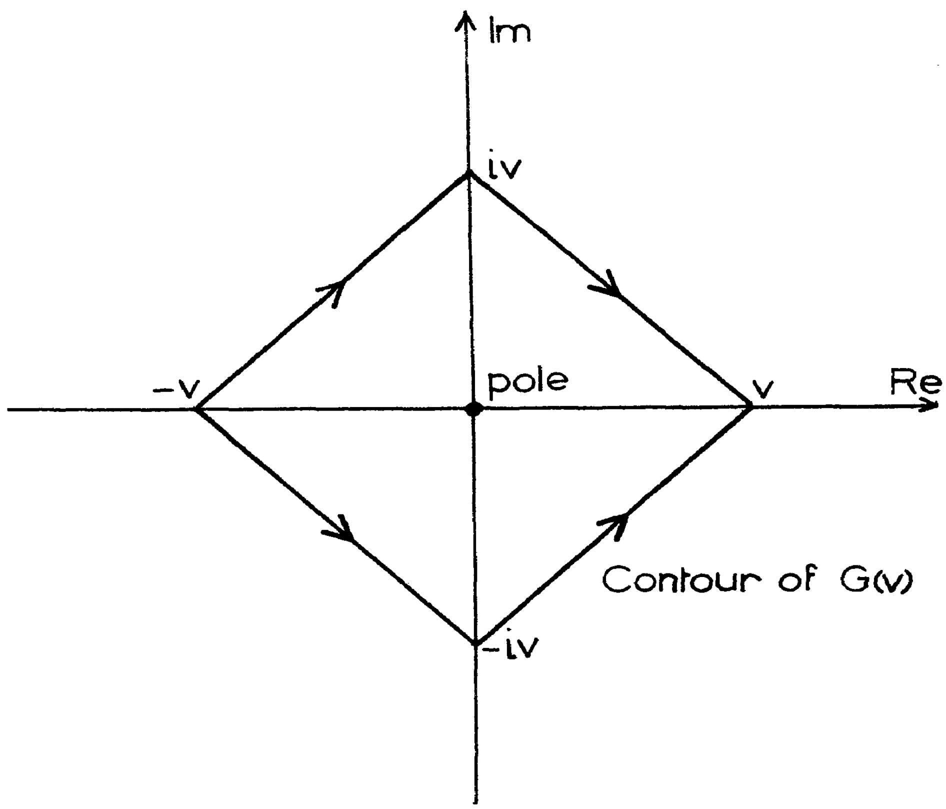

fig 1

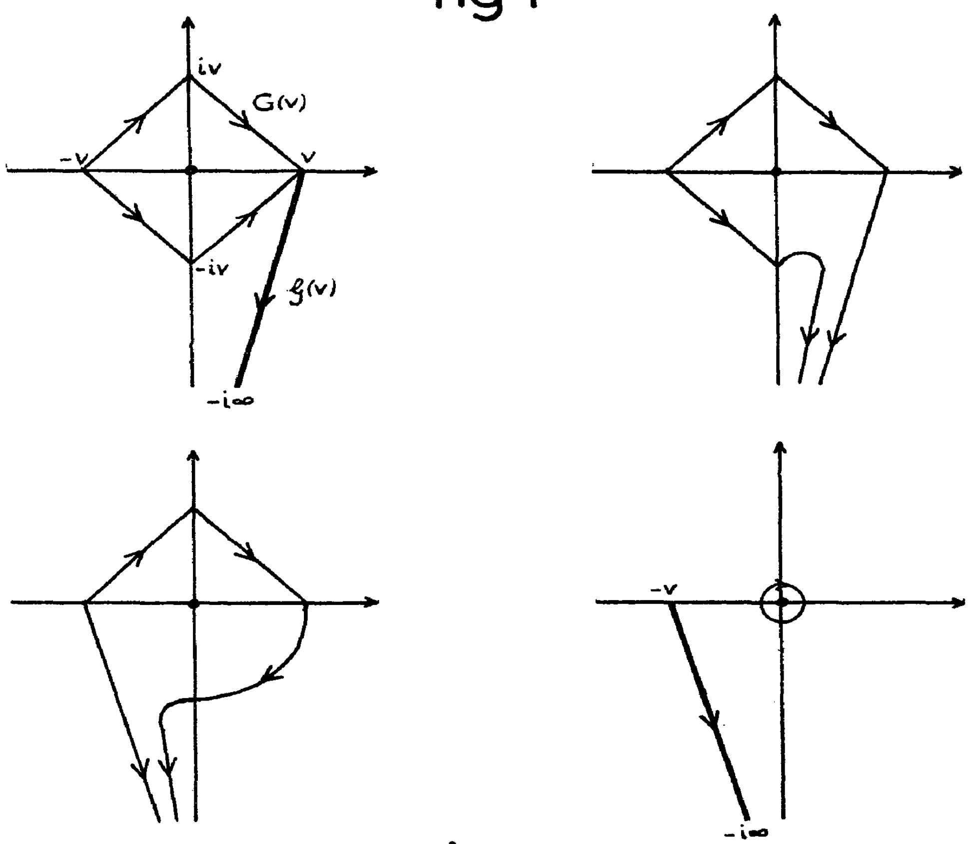

fig 2

Hence

$$G(\nu)+\mathscr{G}(\nu)=-i(2\pi\alpha)^{\frac{1}{2}}\frac{d}{d\nu}\left\{2\int_{-\nu}^{-i\infty}\frac{e^{-i\alpha y}}{y}\eta\left\{\frac{i\alpha}{2y}(y^2-\nu^2)\right\}dy+\right.$$

$$\left.+\oint\frac{e^{-i\alpha y}}{y}\eta\left\{\frac{i\alpha}{2y}(y^2-\nu^2)\right\}dy\right\}, \qquad (16)$$

where $\oint$ is taken to mean a closed line integral round the pole in a clockwise direction. If we define another function, y(v), by

$$r(v)=-i(2\pi\alpha)^{\frac{1}{2}}\frac{d}{d\upsilon}\oint\frac{e^{-i\alpha y}}{y}\eta\left\{\frac{i\alpha}{2y}(y^2-\upsilon^2)\right\}dy, \qquad (17)$$

then

$$G(v) + \mathscr{G}(\upsilon) = y(v) - \mathscr{G}(-\upsilon).$$

(18)

3. A NEW KNOWN FUNCTION

The known function G(v) has rather complex contours. The sum of G(v) and $\mathcal{G}$(v), however, has simpler contours and to introduce these into the integral equation (8) we define a new unknown function, $\mathcal{F}$(v), by

$$\mathcal{F}(v) = F(v) - (G(v) + \mathcal{G}(v)).$$

(19)

The integral equation then becomes

$$\mathcal{F}(\upsilon) = g(\upsilon) - \frac{1}{\pi}\left(\frac{\upsilon}{1+\upsilon}\right)^{\frac{1}{2}} e^{-i\alpha\upsilon}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{\mathcal{F}(t)}{t+\upsilon} e^{-i\alpha\upsilon} dt, \tag{20}$$

where

$$g(\upsilon) = -\mathcal{G}(\upsilon) - \frac{1}{\pi}\left(\frac{\upsilon}{1+\upsilon}\right)^{\frac{1}{2}} e^{-i\alpha\upsilon}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\upsilon}}{t+\upsilon}(G(t)+\mathcal{G}(t))\,dt, \tag{21}$$

The new known function is g(v) which can be expected to be more amenable than G(v) since its contours are simpler.

The substitution of (18) in (21) gives

$$g(\upsilon) = -G(\upsilon) - \frac{1}{\pi}\left(\frac{\upsilon}{1+\upsilon}\right)^{\frac{1}{2}} e^{-i\alpha\upsilon}\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+\upsilon}(r(t) - G(-t))dt. \tag{22}$$

We now consider the various functions involved. From (13) and (15)

$$\mathcal{G}(\upsilon) = -2\upsilon e^{-i\alpha\upsilon}\int_0^{-i\alpha\upsilon} \frac{e^{-idx}}{(x+\upsilon)^2} dx, \tag{23}$$

The integral involving $\mathcal{G}$ (-t) in (22) can then be written

$$\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+\upsilon}\mathcal{G}(-t)\,dt = \int_0^{-i\infty} e^{-idx} \quad 2\int_0^1 \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(t+\upsilon)(x-t)} dt\,dx \tag{24}$$

$$= \pi\left(\frac{1+\upsilon}{\upsilon}\right)^{\frac{1}{2}} e^{i\alpha\upsilon}\mathcal{G}(\upsilon) + \int_0^{\infty} e^{-i\alpha x} \oint \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(t+\upsilon)(x-t)^2} dt\,dx\,,$$

using a result in the appendix.
Now, evaluation of (17) shows that

$$y(v) = 2\pi\alpha v \tag{25}$$

and evaluation of the inner integral in (24) gives

$$\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+\upsilon} \mathscr{G}(-t)\,dt =$$

$$= \pi\left(\frac{1+\upsilon}{\upsilon}\right)^{\frac{1}{2}} e^{i\alpha\upsilon}\mathscr{G}(\upsilon) + \int_0^{i\infty} \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+v} \gamma(t)\,dt \quad . \tag{26}$$

Substitution of (26) in (22) then produces extensive cancellation resulting in a simpler expression for the new known function now given by

$$g(\upsilon) = \frac{1}{\pi}\left(\frac{\upsilon}{1+\upsilon}\right)^{\frac{1}{2}} e^{-i\alpha\upsilon} \int_1^{1+i\infty} \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{e^{-i\alpha t}}{t+\upsilon} \gamma(t)dt. \tag{27}$$

The function g(v) is therefore a straightforward integral with limits which yield a simple asymptotic expansion for large α. y(t) is an integral round a pole, (evaluated in (25)), all other contour integrals having vanished. The cancellations are more extensive than might have been anticipated, and the relatively simple form of g(v) suggests that the contour deformations and the resulting function y(v) may be in some sense fundamental to the problem. Essentially the same known function is derived by Jones [1] without the use of contour deformations. The present derivation is perhaps of more interest in that a deeper insight is given, particularly in respect of the way in which the various functions are associated with each other.

4. THE FIELD ON THB DISC

The function y(v) may be incorporated in Jones' [1] iteration scheme thereby enabling the scheme to start one stage earlier. This illustrates the fundamental nature of y(v). The solution of (20) may then be expressed by

$$\mathcal{F}(\upsilon) \sim \sum_{k=1}^{\infty} \int_0^{\infty} \Psi_k(w) \frac{e^{-i\alpha(w-\upsilon)}}{w-\upsilon} dw \tag{28}$$

where

$$\Psi_0(w) = \frac{i}{\pi} \gamma(w) \tag{29}$$

$$\Psi_{k+1}(v) = \int_1^{1-i\infty} \left(\frac{w-1}{w}\right)^{\frac{1}{2}} \frac{e^{-i\alpha w}}{w+\upsilon} \Psi_k(w) dw \quad , \tag{30}$$

$$\Psi_{k+1}(w) = \frac{i\alpha}{4\pi} \int_0^{\infty} \left(\frac{\upsilon}{1+\upsilon}\right)^{\frac{1}{2}} \Psi_{k+1}(\upsilon) e^{-i\alpha v} M(\upsilon, w) d\upsilon \quad , \tag{31}$$

And

$$M(\upsilon, w) = \frac{4w}{\upsilon^2 - w^2} \left\{ \upsilon H_1^{(2)}(\alpha\upsilon) J_0(\alpha w) - w H_0^{(2)}(\alpha\upsilon) J_1(\alpha w) \right\}.$$

An expression for the field on the disc, f(w), can be found from equations (9), (18), (19) and the iteration scheme above. We may write

$$f(w) = f_0(w) + \sum_{k=1}^{\infty} f_k(w) \tag{32}$$

where

$$f_o(w) = -\frac{e^{i\alpha w}}{\pi^2 w^{\frac{1}{2}} (1-w)^{\frac{1}{2}}} \int_0^1 \left(\frac{1-\upsilon}{\upsilon}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\upsilon}}{\upsilon - w} (\gamma(\upsilon) - \mathcal{G}(-\upsilon)) d\upsilon \quad , \tag{33}$$

and

$$f_k(w) = -\frac{e^{i\alpha w}}{\pi^2 w^{\frac{1}{2}} (1-w)^{\frac{1}{2}}} \int_0^1 \left(\frac{1-\upsilon}{\upsilon}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\upsilon}}{\upsilon - w} \int_0^{\infty} \Psi_k(t) \frac{e^{-i\alpha(t-\upsilon)}}{t-\upsilon} dt \, d\upsilon \, . \tag{34}$$

The term involving g(-v) in (33) is similar to the left hand side of (26). A similar analysis, allowing for the pole at v = w, where $0 \leq w < 1$, then gives

$$\int_0^1 \left(\frac{1-\upsilon}{\upsilon}\right)^{\frac{1}{2}} \frac{e^{-e\alpha\upsilon}}{\upsilon - w} \mathscr{G}(-\upsilon)\, d\upsilon = \pi i \left(\frac{1-w,}{w}\right)^{\frac{1}{2}} e^{-i\alpha w}\gamma(w) +$$

$$+ \int_0^1 \left(\frac{1-\upsilon}{\upsilon}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\upsilon}}{\upsilon - w}\gamma(\upsilon)d\upsilon + 1\int_1^{1-i\infty} \left(\frac{1-\upsilon}{\upsilon}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\upsilon}}{\upsilon - w}\gamma(\upsilon)d\upsilon \ , \tag{35}$$

where the integral on the left hand side and the first integral on the right hand side are principal values. Substitution of this result in (33) shows that

$$f_0(w) = \frac{1}{w}\left\{\frac{i}{\pi}\gamma(w)\right\} + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\int_1^{1-i\infty} \left(\frac{\upsilon-1}{\upsilon}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\upsilon}}{\upsilon - w}\left\{\frac{i}{\pi}\gamma(\upsilon)\right\}d\upsilon$$

$$= \frac{1}{w}\Psi_0(w) + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\psi_1(-w). \tag{36}$$

In considering the expression (34) for f (w) we change the order of integration.

Since

$$\int_0^1 \left(\frac{1-\upsilon}{\upsilon}\right)^{\frac{1}{2}} \frac{d\upsilon}{(\upsilon-w)(\upsilon-t)} = \pi^2\left(\frac{1-w}{w}\right)^{\frac{1}{2}}\delta(t-w) + \pi\left(\frac{t-1}{t}\right)^{\frac{1}{2}}\frac{H(t-1)}{t-w},$$

$0 < w < 1$, $t > 0$, (Jones [1],appendix B),

we can write

$$f_k(w) = \frac{1}{w}\Psi_k(w) + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\Psi_{k+1}(-w). \tag{37}$$

The expression for $f_0(w)$ and $f_k(w)$, $k \geq 1$, given by (36) and (37) are now of the same form, and so

$$f(w) \sim \sum_{k=0}^{\infty} f_k(w), \tag{38}$$

and

$$f_k(w) = \frac{1}{w}\Psi_k(w) + \frac{\Psi_{k+1}(-w)e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \tag{39}$$

5. THE FAR SCATTERED FIELD.

The scattered field $u_S(\underset{\sim}{R})$ is given by equation (2). In general the integral is complicated. However, if $\underset{\sim}{R}$ is assumed to be far from the disc a useful approximation can be made. Then, using spherical polar coordinates,

$$U_S\ (\underset{\sim}{R}) \sim -\frac{1}{2}\frac{e^{-i\alpha R}}{R}\int_0^1 wf(w)j_0\ (\alpha w \sin\theta)dw\ + O\left(\frac{1}{R^2}\right) \tag{40}$$

Let

$$F(\theta) = \int_0^1 wf(w)\ J_0\ (\alpha\ w\ \mathit{sin}\ \theta)\ dw \tag{41}$$

so that

$$U_S\ (\underset{\sim}{R}) \sim -\frac{1}{2}\frac{e^{-i\alpha R}}{R}\ F(\theta) \tag{42}$$

Then, by (38),

$$F(\theta) \sim \sum_{k=0}^{\infty} F_k\ (\theta)\quad , \tag{43}$$

where

$$\begin{aligned} F_k(\theta) &= \int_0^1 wf_k(w)\ J_0\ (\alpha\, w\, \mathit{sin}\,\theta)\ dw \\ &= \int_0^1 \psi_k(w)J_0(\alpha w \sin\theta)dw + \\ &\quad + \frac{1}{\pi}\int_0^1\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\Psi_{k+1}(-w)e^{i\alpha w}J_0(\alpha w\sin\theta)dw \end{aligned} \tag{44}$$

after substituting for $f_k(w)$ from (39).

$F(\theta)$ can be written in a modified form which is more convenient to use. To do this it is necessary to return to the iteration scheme. Equation (31) arises from the solution of an integral equation (Jones [1])

$$\int_0^w \Psi_k(w)\left\{\frac{e^{-i\alpha(w-\upsilon)}}{w-\upsilon} + \frac{e^{i\alpha(e+\upsilon)}}{w+\upsilon}\right\}\ dw = \left(\frac{\upsilon}{1+\upsilon}\right)^{\frac{1}{2}}\Psi_k(\upsilon)e^{-i\alpha\upsilon}$$

This equation, among others, has been considered by Jones in [4] and [5] Multiply through by Jo(αvsinθ) and integrate from 0 to ∞. Then, after reversing the order of integration on the left hand side and changing the sign of v in the second term of the integrand, we have

$$\int_0^{\infty} \Psi_k(w)\, e^{-i\alpha w} \int_{-\infty}^{\infty} \frac{e^{i\alpha\upsilon}}{w-\upsilon} J_0(\alpha\upsilon \sin\theta)\, d\upsilon\, dw \;=$$

$$= \int_0^{\infty} \left(\frac{\upsilon}{1+\upsilon}\right)^{\frac{1}{2}} \Psi_k(\upsilon)\, e^{-i\alpha\upsilon}\, J_0(\alpha\upsilon \sin\theta)\, d\upsilon\ .$$

If the inner integral is now completed with an indentation above the pole at v = w its contour may be deformed upwards without contribution, provided | sin θ | <1. Hence

$$\int_0^{\infty} \Psi_k(w)\, J_0\, (\alpha\ w \sin\theta\,)\, dw \;=$$

$$= \frac{1}{\pi} \int_0^{\infty} \left(\frac{\upsilon}{1+\upsilon}\right)^{\frac{1}{2}} \psi_k(\upsilon)\; e^{-i\alpha\upsilon} J_0\, (\alpha\,\upsilon \sin\theta)\, d\upsilon\ . \qquad (45)$$

(The author is indebted to Professor W.E.Williams for a shortened proof of this result).

Now, considering the right hand side of (44),

$$\int_0^{1} \Psi_k\, (w)\, J_0\, (\alpha\, w \sin\theta)\;\, dw \;=$$

$$= \int_0^{\infty} \Psi_k(w) J_0\, (\alpha w \sin\theta) dw \;-\; \int_0^{\infty} \Psi_k\, (w)\, J_0\, (\alpha w \sin\theta)\;\, dw$$

$$= \frac{i}{\pi} \int_0^{\infty} \left(\frac{\upsilon}{1+\upsilon}\right)^{\frac{1}{2}} \psi_k(\upsilon) e^{-i\alpha\upsilon} J_0(\alpha\upsilon \sin\theta)\, d\upsilon \;-$$

$$- \int_1^{\infty} \Psi_k(w) J_0(\alpha w \sin\theta)\, dw\ ,$$

after using (45). The second term on the right hand side of (46)

(44) may be written,

$$\frac{1}{\pi}\int_0^1 \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w)\, e^{i\alpha w}\, J_0(\alpha w \sin\theta)\, dw \;=$$

$$= \frac{1}{\pi}\int_1^{i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w)\, e^{i\alpha w}\, J_0(\alpha w \sin\theta)\, dw \;-$$

$$- \frac{1}{\pi}\int_{,}^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}} \Psi_{k+1}(-w) e^{i\alpha w}\, J_0(\alpha w \sin\theta)\, dw$$

$$= \; -\frac{i}{\pi}\int_0^{\infty}\left(\frac{w}{1+w}\right)^{\frac{1}{2}} \psi_{k+1}(w)\, e^{-i\alpha w} J_0(\alpha w \sin\theta)\, dw \;-$$

$$- \;\frac{1}{\pi}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w) e^{i\alpha w} J_0(\alpha w \sin\theta)\, dw \quad , \qquad (47)$$

after deforming the contour of the first integral of the right hand side onto the negative real axis and then changing the sign of w.

Comparison of (46) with (47) shows the similarity between the first terms on the right hand side of each equation. Substitution of these two relationships back into (44), the expression for $F_k(\theta)$, shows that cancellation will occur down the iteration scheme. Hence,

$$F(\theta) \sim \sum_{k=0}^{\infty} F_k(\theta)$$

$$\sim \int_0^1 \Psi_0(w)\, J_0\, (\alpha w \sin\theta)\, dw \;-$$

$$- \frac{1}{\pi}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_1(-w) e^{i\alpha w} J_0(\alpha w \sin\theta) dw \;-$$

$$- \sum_{k=1}^{\infty} \left\{ \int_1^{\infty} \psi_k(w) J_0(\alpha w \sin\theta)\, dw \; + \right.$$

$$\left. + \int_1^{1+iw}\left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w)\, e^{i\alpha w} J_0\, (\alpha w \sin\theta)\, dw \right\}. \qquad (48)$$

It is therefore apparent that there is no contribution to the far field from the centre of the disc, except perhaps from the first term. However,

$$\int_0^1 \Psi_0(w) J_0\,(\alpha\, w \sin\theta)\, dw \;=\; \frac{2i}{\sin\theta} J_1(\alpha \sin\theta)\;, \tag{49}$$

there being no contribution from the lower limit. The far field as a whole is therefore determined entirely from conditions at the edge of the disc.

The first two integrals in (48) will be called the aero order contribution to $F(\theta)$ and subsequent pairs of integrals for each k will be called the k^{th} order contributions.

The only restriction which applies to $F(\theta)$ so far is that $|\sin\theta| < 1$, (c.f.(45)). This means that the expression cannot be used to find the field in the plane of the disc.

In order to proceed to the asymptotic evaluation of the second and subsequent integrals in (48) it is necessary to consider three regions in which the field may be required. The first region is that at or near the axis of symmetry, such that $|\alpha \sin\theta| << 1$. An important particular case of this is the far field along the axis of symmetry, which gives the scattering cross section of the disc (Jones [1]). The field in this region may be determined by using the series forms of the Bessel functions. The second region is that in which $\alpha \sin\theta$ is neither very small nor very large. This requires special attention as a transition region, and will not be considered here. The third region is that in which $|\alpha \sin\theta| >> 1$, and at high frequencies this is by far the largest region. In this region the asymptotic forms of the Bessel functions may be used in the integrals. This third region is the one considered in the present work.

6. THE ZERO ORDER CONTRIBTPION TO F(θ).

In order to find the far field it is necessary to calculate F(θ) in. detail. The zero order contribution is given by

$$\int_0^1 \Psi_0(w) J_0(\alpha w \sin\theta)\, dw \; -$$

$$- \; \frac{1}{\pi}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_1(-w) e^{i\alpha w} J_0(\alpha w \sin\theta)\, dw. \qquad (50)$$

At high frequencies, away from the axis of symmetry, $|\alpha \sin\theta| >> 1$ and an asymptotic form of the Bessel function may be employed. The particular form used is

$$J_0(z) \sim$$

$$\sim \frac{1}{\pi^{\frac{1}{2}}} e^{-i\left(\upsilon\frac{\pi}{3}+\frac{\pi}{4}\right)} \sum_{m=0}^{\infty} \frac{i^m(\upsilon+m-\frac{1}{2})!}{m!(\upsilon-m-\frac{1}{2})!} \frac{1}{(2z)^{m+\frac{1}{2}}}\left(e^{iz} + i(-1)^{m+\upsilon} e^{iz}\right). \qquad (51)$$

The first integral of the zero order field has been found in (49) and 30, using (51),

$$\int_0^1 \Psi_0(w) J_0(\alpha w \sin\theta)\, dw \; \sim$$

$$\sim \; - \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \alpha \left(\frac{i}{\alpha\sin\theta}\right)^{\frac{3}{2}} \Big\{ (e^{i\alpha\sin\theta} - i e^{-i\alpha\sin\theta}) \; +$$

$$+ \; \frac{3i}{8\alpha\sin\theta}\left(e^{i\alpha\sin\theta} + ie^{-i\alpha\sin\theta}\right) \; +$$

$$+ \; \frac{15}{2^7\alpha^2\sin^2\theta}\left(e^{i\alpha\sin\theta} - ie^{-i\alpha\sin\theta}\right)\Big\} + O(\alpha^{-\frac{7}{2}}) \; . \qquad (52)$$

The second integral of the zero order field may be written

$$\frac{1}{\pi}\int_1^{1-i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi(-w)\, e^{i\alpha w} J_0(\alpha w \mathit{sin}\, \theta)\, dw$$

$$= \; \frac{1}{\pi}\int_1^{1-i\infty}\left(\frac{t-1}{t}\right)^{\frac{1}{2}} \Psi_0(t) e^{-i\alpha t}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}} \frac{J_0(\alpha w \sin\theta)}{t-w} e^{i\alpha w}\, dwdt. \qquad (53)$$

Use of the asymptotic form of the Bessel function and some fairly straightforward analysis shows that the inner of these integrals may be written

$$\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\frac{J_0(\alpha w\sin\theta)}{t-w}\,e^{i\alpha w}dw \sim$$

$$\sim -\frac{i}{\pi^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{i^m\,(m-\frac{1}{2})!}{m!(-m-\frac{1}{2})!}\,\frac{1}{(2\alpha\sin\theta)^{m+\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{(-m)!\,(n-\frac{1}{2})!\,i^n}{n!\,(-m-n)!}\times$$

$$\times\left\{e^{i\alpha t(1+\sin\theta)}\int_{\alpha(1+\sin\theta)}^{\infty}\frac{e^{-i(t-1)x}}{x^{n+\frac{1}{2}}}\,dx\;+\right.$$

$$\left.+\;i(-1)^m e^{i\alpha t(1-\sin\theta)}\int_{\alpha(1-\sin\theta)}^{\infty}\frac{e^{-i(t-1)x}}{x^{n+\frac{1}{2}}}\,dx\right\}, \qquad (54)$$

Substitution of this result in (53) and inversion of the order of integration yields integrals of the form,

$$\int_{\alpha(1\pm\sin\theta)}^{\infty}\frac{e^{ix}}{x^{n+\frac{1}{2}}}\int_1^{1-i\infty}\left(\frac{t-1}{t}\right)^{\frac{1}{2}}\Psi(t)e^{-it(x\mp\alpha\sin\theta)}\,dt\,dx\;.$$

After substituting into these integrals the expression for $\Psi_0(t)$ from (29) they may be asymptotically evaluated to complete the evaluation of (53). Hence

$$\frac{1}{\pi}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\psi_1(-w)e^{i\infty w}J_0(\infty w\sin\theta)\;dw\;\sim$$

$$\sim -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\alpha\left(\frac{i}{\alpha\sin\theta}\right)^{\frac{3}{2}}\left\{\left[\left[\left\{1-(1+\sin\theta i^{\frac{1}{2}}\right\}e^{i\alpha\sin\theta}-i\{1-(1-\sin\theta i^{\frac{1}{2}}\}e^{-i\alpha\sin\theta}\right]\right.\right.$$

$$+\frac{i}{\alpha\sin\theta}\left[\frac{5}{8}[\{1-(1+\sin\theta i^{\frac{1}{2}}\}e^{i\alpha\sin\theta}+i\{1-(1-\sin\theta i^{\frac{1}{2}}\}e^{-i\alpha\sin\theta}]\;-\right.$$

$$\left.-\frac{1}{4}[\{1-(1+\sin\theta i^{\frac{3}{2}}\}e^{i\alpha\sin\theta}+i\{1-(1-\sin\theta i^{\frac{3}{2}}\}e^{-i\alpha\sin\theta}]\right]$$

$$+\left(\frac{i}{\alpha\sin\theta}\right)^2\left[\frac{1}{16}[\{1-(1+\sin\theta)^{-\frac{1}{2}}\}e^{i\alpha\sin\theta}-i\{1-(1-\sin\theta)^{-\frac{1}{2}}\}e^{-i\alpha\sin\theta}]\;-\right.$$

$$-\frac{55}{128}[\{1-(1+\sin\theta i^{-\frac{1}{2}}\}e^{i\alpha\sin\theta}-i\{1-(1-\sin\theta)^{\frac{1}{2}}\}e^{-i\alpha\sin\theta}]\;+$$

$$+\frac{11}{321}[\{1-(1+\sin\theta)^{\frac{3}{2}}\}e^{i\alpha\sin\theta}]-i\{1-(1-\sin\theta)^{\frac{3}{2}}\}e^{-i\alpha\sin\theta}]\;-$$

$$+O\left(\alpha^{-\frac{7}{2}}\right)\quad. \qquad (55)$$

It is noted that the terms of (52), the first integral in the zero order contribution, correspond exactly with those terms or {55) coining from the first 1 of $\{1 - (1 \pm \sin\theta)^{r+\frac{1}{2}}\}$. In the zero order contribution as a whole there is complete cancellation of the first integral with part of the second integral. The zero order contributions to F(θ) is therefore,

$$\int_0^1 \Psi(w)\, J_0(\alpha\, w \sin\theta)\, dw \; -$$

$$- \frac{1}{\pi}\int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_1(-w)\, J_0(\alpha\, w \sin\theta)\, dw \sim$$

$$\sim \left(\frac{2}{\pi\alpha}\right)^{\frac{1}{2}} \frac{e^{-i\frac{\pi}{4}}}{(\sin\theta)^{\frac{3}{2}}} \Bigg\{ [\,(1+\sin\theta)^{\frac{1}{2}} e^{i\alpha\sin\theta} - i\,(1-\sin\theta)^{\frac{1}{2}} e^{-i\alpha\sin\theta}] +$$

$$+ \frac{i}{\alpha\sin\theta} \left(\frac{5}{8} [\,(1+\sin\theta)^{\frac{1}{2}} e^{i\alpha\sin\theta} + i(1-\sin\theta)^{\frac{1}{2}} e^{i\alpha\sin\theta}\,] - \right.$$

$$\left. - \frac{1}{4} [\,(1+\sin\theta)^{\frac{3}{2}} e^{\,i\,\alpha\,\sin\theta} + i(1-\sin\theta)^{\frac{1}{2}} e^{-i\alpha\sin\theta}\,] \right) -$$

$$- \left(\frac{i}{\alpha\sin\theta}\right)^2 \left(\frac{1}{16} [\,(1+\sin\theta)^{-\frac{1}{2}} e^{i\,\alpha\,\sin\theta} - i\,(1-\sin\theta)^{-\frac{1}{2}} e^{-i\,\alpha\,\sin\theta}\,] - \right.$$

$$- \frac{55}{128} \left[(1+\sin\theta)^{\frac{1}{2}} e^{i\alpha\sin\theta} - i\,(1-\sin\theta)^{\frac{1}{2}} e^{-i\alpha\sin\theta} \right] +$$

$$+ \frac{11}{32} \left[(1+\sin\theta)^{\frac{3}{2}} e^{i\alpha\sin\theta} - i\,(1-\sin\theta)^{\frac{3}{2}} e^{-i\alpha\sin\theta} \right] -$$

$$\left. - \frac{3}{32} \left[(1+\sin\theta)^{\frac{5}{2}} e^{i\,\alpha\,\sin\theta} - i\,(1-\sin\theta)^{\frac{5}{2}} e^{-i\,\alpha\,\sin\theta} \right] \right) \Bigg\}$$

$$+ O\left(\alpha^{-\frac{7}{2}}\right) \quad . \tag{56}$$

7. AN ASYMPTOTIC FORM OF $\Psi_k(w)$

The k[th] order contribution to F(θ) is given by

$$- \int_1^{\infty} \Psi_k(w)\, J_0(\alpha\, w \sin\theta\,)\, dw \quad -$$

$$- \frac{1}{\pi} \int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w)\, e^{i\,\alpha\, w} J_0\,(\alpha\, w \sin\theta)\; dw \;.$$

To evaluate these integrals it is necessary to consider a suitable form or $\Psi_k(w)$. It is noted that in the first integral $w \geq 1$. In the second integral $\psi_{k+1}(-w)$ occurs, which may be expressed as an integral of Ψ_k from 1 to 1- i∞ by (30). In each case the modulus of the argument of Ψ_k is greater than or equal to unity. This knowledge may be used to find an asymptotic form of $\Psi_k(w)$ From (31)

$$\Psi_k(w) \;=\; \frac{i\alpha}{4\pi}\int_0^{\infty}\left(\frac{\upsilon}{1+\upsilon}\right)^{\frac{1}{2}} \psi_k(\upsilon) e^{-i\alpha\upsilon}\; M(\upsilon, w)\, d\upsilon$$

Where

$$M\,(\upsilon, w) \;=\; \frac{4w}{\upsilon^2 - w^2}\left\{\upsilon H_1^{(2)}(\alpha w) j_0(\alpha w) - w H_0^{(2)}(\alpha\upsilon) j_1\,(\alpha w)\right\}.$$

The function M(v,w) has poles at v=± w, and so $\Psi_k(w)$ is a principal value. It may be assumed (Jones [1]) that $\Psi_k(v)$ is regular in the region Re(v) > - 1 of the complex v plane and is bounded as $|\ v\ | \rightarrow \infty$ in this region. Completion of the contour of integration with an indentation below the pole at v = w gives,

$$\Psi_k(w) \;=\; \frac{1\alpha}{4\pi}\int_0^{\infty}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} \psi_k\;(v)\; e^{-i\alpha w} M\,(v, w)\; dv \;+$$

$$+\; \frac{i}{\pi}\left(\frac{w}{1+w}\right)^{\frac{1}{2}} \psi_k(w) e^{-i\alpha w}.$$

The contour can now be deformed onto the negative imaginary axis. On changing v to -iv we obtain,

$$\Psi_k(w) \sim \frac{i}{\pi}\left(\frac{w}{1+w}\right)^{\frac{1}{2}} \psi_k(w)\, e^{-i\alpha w} -$$

$$- \frac{2\alpha}{\pi^2}\, e^{i\frac{\pi}{4}} \int_0^\infty \left(\frac{v}{1-iv}\right)^{\frac{1}{2}} \psi_k(-iv) e^{-\alpha v} \times$$

$$\times \frac{w}{v^2+w^2} \left\{ v K_1(\alpha v) J_0(\alpha w) - w\, K_0(\alpha v) J_1\ (\alpha w)\right\} dv$$

$$\sim \frac{i}{\pi}\left(\frac{w}{1+w}\right)^{\frac{1}{2}} \psi_k(w) e^{-i\alpha w} - \frac{2\alpha}{\pi^{\frac{3}{2}}}\ e^{i\frac{\pi}{4}} \sum_{m=0}^{\infty} \frac{\left(-\frac{1}{2}\right)!\left(-i\right)^m}{M!\left(-\frac{1}{2}-m\right)!} \times$$

$$\times \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\ \psi_k^{(n)}(0) \sum_{p=0}^{\infty} \frac{\left(m+n+2p+\frac{1}{2}\right)!\ \left(-1\right)^p}{(2\alpha)^{m+n+2p+\frac{3}{2}}} \times$$

$$\times \left\{ \frac{\left(m+n+2+-\frac{5}{2}\right)!}{2\alpha\left(m+n+2p+2\right)!}\ \frac{J_0(\alpha w)}{w^{2p+1}} - \frac{\left(m+n+2p++\frac{1}{2}\right)!}{\left(m+n+2p++1\right)!}\ \frac{J_1(\alpha w)}{w^{2p}} \right\},$$

after asymptotic evaluation of the integral. This form of $\Psi_k(w)$ is suitable where $|\ w\ | \geq 1$.

8. THE k^{th} ORDER CONTRIBUTION TO F(θ).

It is now possible to proceed with a detailed calculation of the k order field which is given by the two integrals

$$-\int_1^\infty \Psi_k(w) j_0(\alpha\, w \sin\theta) dw \;-$$
$$-\;\frac{1}{\pi}\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_{k+1}(-w)\, e^{i\alpha w} J_0(\alpha\; w\; \sin\;\theta)\;\; dw\;\; . \qquad (58)$$

In considering the first integral the asymptotic form of $\Psi_k(w)$ can he employed since $w \geq 1$. The resulting integrals may then be evaluated asymptotically after some straightforward but tedious analysis to give,

$$\int_1^\infty \Psi_k(w)\; J_0\;(\alpha w \sin\theta)\; dw \;\sim$$
$$\sim \frac{e^{-i\left(\alpha+\frac{\pi}{4}\right)}}{2(\alpha\pi)^{\frac{3}{2}}(\sin\theta)^{\frac{1}{2}}}\left\{\psi_k(1)\left[\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}+i\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right] -\right.$$
$$-\;\frac{i}{\alpha}\left(\left[\;\psi'_k(1)-\frac{1}{4}\psi_k(1)\right]\left[\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}+i\;\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\right]+\right.$$
$$\left.\left.+\;\frac{\psi_k(1)}{8\sin\theta}\left[\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)}\;-\;i\;\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)}\right]\right)\right\}\;-$$
$$-\frac{e^{i\frac{3\pi}{4}}}{2^{7/2}\pi^{3/2}\alpha^{5/2}(\sin\theta)^{\frac{1}{2}}}\;\psi_k(O)\left\{\frac{e^{i\alpha(1+\sin\theta)}-e^{-i\alpha(1+\sin\theta)}}{1+\sin\theta}\right.\;+$$
$$\left.+\;i\;\frac{e^{i\alpha(1-\sin\theta)}\;+\;e^{-i\alpha(1-\sin\theta)}}{1-\sin\theta}\right\}\;+\;O\left(\frac{\psi_k}{\alpha^{\frac{7}{2}}}\right). \qquad (59)$$

The second integral in (58) can also be evaluated

asymptotically though the labour involved is fairly extensive.

The final result is

$$\frac{1}{\pi}\int_{1}^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\psi_{k+1}(-w)e^{i\alpha w}\,j_0(\alpha w\sin\theta)\,dw \;\sim$$

$$\sim\; -\;\frac{e^{-i\left(\alpha+\frac{\pi}{4}\right)}}{(\pi\alpha)^{\frac{3}{2}}(\sin\theta)^{\frac{1}{2}}}\left\{\;\frac{1}{2}\psi_k(1)\left[\;\left\{1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{-i\alpha\sin\theta}}{1-\sin\theta}\;+\right.\right.$$

$$\left.+\;i\left\{\;1-\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\;\right]-$$

$$-\;\frac{i}{\alpha}\left(\frac{3}{4}\left(\psi_k'(1)-\frac{1}{4}\psi_k(1)\right)\left(\left[\;\left\{1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}\;+\right.\right.\right.$$

$$\left.+\;i\left\{1-\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{-i\alpha\sin\theta}}{(1-\sin\theta)^2}\right]-$$

$$\left.\left.-\;\frac{1}{3}\left[\;\left\{\;1-\;\left(\frac{1+\sin\theta}{2}\right)^{\frac{3}{2}}\right\}\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}\;+\;i\;\left\{1-\;\left(\frac{1-\sin\theta}{2}\right)^{\frac{3}{2}}\right\}\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\right]\right)+\right.$$

$$\left.\left.+\;\frac{\psi_k(1)}{2^4\sin\theta}\left[\;\left\{1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}\;-i\left\{\;1-\left(\frac{1-\sin\theta}{2}\right)^{\frac{3}{2}}\right\}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\;\right]\right)\right\}+$$

$$+\;\frac{\psi_k(0)e^{i\alpha}}{2^{\frac{7}{2}}\pi\alpha^2(\sin\theta)^{\frac{1}{2}}}\left[\;\frac{e^{i\alpha\alpha\sin}}{(1+\sin\theta)^{\frac{1}{2}}}\;+\;i\;\frac{e^{-i\alpha\alpha\sin}}{(1-\sin\theta)^{\frac{1}{2}}}\right]+$$

$$+\;\frac{\psi_k(0)e^{i\left(\alpha+\frac{3\pi}{4}\right)}}{2^{\frac{7}{2}}\pi^{\frac{3}{2}}\alpha^{\frac{5}{2}}(\sin\theta)^{\frac{1}{2}}}\left[\frac{e^{i\alpha\alpha\sin}}{1+\sin\theta}\;+\;\frac{e^{-i\alpha\alpha\sin}}{1-\sin\theta}\;\right]\;-$$

$$-\;\frac{\psi_k(0)e^{-i\left(\alpha-\frac{\pi}{4}\right)}}{2^{\frac{7}{2}}\pi^{\frac{3}{2}}\alpha^{\frac{5}{2}}(\sin\theta)^{\frac{1}{2}}}\left[\;\left\{\;1-\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{-\alpha\sin\theta}}{1-\sin\theta}\;+\right.$$

$$\left.+\;i\left\{1-\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\right\}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\;\right]+$$

$$+\;O\left(\frac{\psi_k}{\alpha^3}\right). \tag{60}$$

When the two integrals of the k^{th} order contribution are combined, the first integral cancels completely with part of the second in a similar manner to the cancellation which occurred in the zero order contribution to F(θ). An asymptotic development of the kth order contribution is therefore given by

$$
\begin{aligned}
&-\int_1^{\infty}\psi_k(w)\,J_0(\alpha w\sin\theta)dw - \\
&\quad -\int_1^{1+i\infty}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\psi_{k+1}(-w)\,e^{i\alpha w}J_0(\alpha w\sin\theta)\,dw \sim \\
&\sim \frac{e^{-i\left(\alpha+\frac{\pi}{4}\right)}}{(\pi\alpha)^{\frac{3}{2}}(\sin\theta)^{\frac{1}{2}}}\Bigg\{-\frac{1}{2}\psi_k(1)\Bigg[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}+ \\
&\qquad\qquad + i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\Bigg]+ \\
&+\frac{i}{\alpha}\Bigg(\frac{3}{4}\left(\psi_k^{|}(1)-\frac{1}{4}\psi_k(1)\right)\Bigg(\Bigg[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{(1+\sin\theta)^2}+ \\
&\qquad\qquad + i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\Bigg]- \\
&\quad -\frac{1}{3}\Bigg[\left(\frac{1+\sin\theta}{2}\right)^{\frac{3}{2}}\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}+i\left(\frac{1-\sin\theta}{2}\right)^{\frac{3}{2}}\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\Bigg]\Bigg)- \\
&\quad -\frac{\psi_k(1)}{2^4\sin\theta}\Bigg[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}-i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\Bigg]\Bigg)\Bigg\}- \\
&\quad -\frac{\psi_k(0)e^{i\alpha}}{2^{\frac{7}{2}}\pi\alpha^2(\sin\theta)^{\frac{1}{2}}}\Bigg[\frac{e^{i\alpha\sin\theta}}{(1+\sin\theta)^{\frac{1}{2}}}+i\frac{e^{-i\alpha\sin\theta}}{(1-\sin\theta)^{\frac{1}{2}}}\Bigg]- \\
&\quad -\frac{\psi_k(0)e^{-i\left(\alpha-\frac{\pi}{4}\right)}}{2^{\frac{7}{2}}\pi^{\frac{3}{2}}\alpha^{\frac{5}{2}}(\sin\theta)^{\frac{1}{2}}}\Bigg[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}+ \\
&\qquad\qquad + i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\Bigg]+ \\
&\quad + O\left(\frac{\psi_k}{\alpha^3}\right). \qquad (61)
\end{aligned}
$$

9. THE FIRST AND SECOMD ORDER CONTRIBUTIONS

An integral form of the far field has been derived in section 5 and the integrals involved considered in sections 6, 7 and 8. The zero order contribution is given by (56) and the general k^{th} order contribution by (61). This last equation, however, is in terms of ψ_k and its derivatives and so expansions of these must be found.

From equation (30) in the iteration scheme

$$\psi_{k+1}(\upsilon) = \int_1^{1-i\infty} \left(\frac{w-1}{w}\right)^{\frac{1}{2}} \frac{e^{-i\alpha w}}{w+\upsilon} \psi_k(w)dw. \tag{62}$$

The function $\psi_1(\upsilon)$ therefore involves $\psi_0(w)$ which is known. Asymptotic evaluation of the integral then gives,

$$\psi_1(\upsilon) \sim \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{-i\left(\alpha+\frac{\pi}{4}\right)} \left\{ \frac{1}{1+\upsilon} + \frac{3i}{2\alpha}\left[\frac{1}{(1+\upsilon)^2} - \frac{1}{2(1+\upsilon)}\right] - \right.$$

$$\left. - \frac{15}{4\alpha^2}\left[\frac{1}{(1+\upsilon)^3} - \frac{1}{2(1+\upsilon)^2} - \frac{1}{8(1+\upsilon)}\right]\right\} +$$

$$+ O\left(\alpha^{-\frac{7}{2}}\right) \tag{63}$$

Hence,

$$\psi_1(1) \sim \frac{1}{2}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{-i\left(\alpha+\frac{\pi}{4}\right)} \left\{1 + \frac{15}{32\alpha^2}\right\} + O\left(\alpha^{-\frac{7}{2}}\right),$$

$$\psi_1(1) \sim -\frac{1}{4}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{-i\left(\alpha+\frac{\pi}{4}\right)} \left\{1 + \frac{3i}{4\alpha} - \frac{15}{32\alpha^2}\right\} + O\left(\alpha^{-\frac{7}{2}}\right),$$

$$\psi_1(0) \sim \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{-i\left(\alpha+\frac{\pi}{4}\right)} \left\{1 + \frac{3i}{4\alpha} - \frac{45}{32\alpha^2}\right\} + O\left(\alpha^{-\frac{7}{2}}\right),$$

$$\psi_1^{|}(0) \sim -\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{-i\left(\alpha+\frac{\pi}{4}\right)} \left\{1 + \frac{9i}{4\alpha} - \frac{225}{32\alpha^2}\right\} + O\left(\alpha^{-\frac{7}{2}}\right),$$

Equation (62) shows that $\psi_{k+1}(\upsilon)$ is an integral of $\psi_k(w)$ and the range of integration is such that $|w| \geq 1$. The asymptotic form of $\psi_k(w)$ may therefore be used and so $\psi_{k+1}(\upsilon)$ becomes a function of ψ_k and its derivatives. Some fairly straightforward analysis shows that

$$\psi_{k+1}(\upsilon) \sim \frac{i\psi_k(0)}{2^3\alpha}\frac{1}{\upsilon}\left(1-(1+\upsilon)^{\frac{1}{2}}\right)+\frac{e^{-i\left(2\alpha+\frac{\pi}{4}\right)}}{2^3\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\frac{\psi_k(1)}{1+\upsilon}+$$
$$+\frac{9}{2^7\alpha^2}\left\{\left(\psi'_k(0)-\frac{1}{2}\psi_k(0)\right)\frac{1}{u}\left(1-(1+\upsilon)^{\frac{1}{2}}\right)-\right.$$
$$\left.-\psi_k(0)\frac{1}{\upsilon^2}\left(1+\frac{1}{2}\upsilon-(1+\upsilon)^{\frac{1}{2}}\right)\right\}+O\left(\frac{\psi_k}{\alpha^{\frac{5}{2}}}\right). \qquad (64)$$

Use of this result and the expressions for ψ_1 above give the following expressions for ψ_2,

$$\psi_2(1) \sim -\frac{(2^{\frac{1}{2}}-1)}{2^3\alpha^{\frac{3}{2}}}\pi^{\frac{1}{2}}e^{-i(\alpha-\frac{\pi}{4})}-\frac{ie^{-3i\alpha}}{2^5\alpha^2}+O(\alpha^{-\frac{5}{2}}),$$

$$\psi_2(0) \sim -\frac{\pi^{\frac{1}{2}}}{2^3\alpha^{\frac{3}{2}}}e^{-i(\alpha-\frac{\pi}{4})}-\frac{ie^{-3i\alpha}}{2^4\alpha^2}+O(\alpha^{-\frac{5}{2}}),$$

If the expressions for ψ_1, and ψ'_1, are now substited into (61) we have, for the first order contribution to $f(\theta)$,

$$-\int_1^{\infty}\psi_1(w)J_0(\alpha w\sin\theta)dw-\int_1^{1+i\alpha}\left(\frac{w}{1-w}\right)^{\frac{1}{2}}\psi_2(-w)e^{i\alpha w}J_0(\alpha w\sin\theta)dw$$
$$\sim\frac{ie^{-2i\alpha s}}{\pi\alpha^2(\sin\theta)^{\frac{1}{2}}}\left\{\frac{1}{4}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}+i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right]\right.$$
$$+\frac{i}{\alpha}\left(\frac{9}{32}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{3}{2}}\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}+i\left(\frac{1-\sin\theta}{2}\right)^{\frac{3}{2}}\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\right]-\right.$$
$$-\frac{3}{32}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{3}{2}}\frac{e^{i\alpha\sin\theta}}{(1-\sin\theta)^2}+i\left(\frac{1-\sin\theta}{2}\right)^{\frac{3}{2}}\frac{e^{-i\alpha\sin\theta}}{(1+\sin\theta)^2}\right]+$$
$$\left.\left.+\frac{1}{32\sin\theta}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}-i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right]\right)\right\}$$
$$-\frac{e^{-i\frac{\pi}{4}}}{2^{\frac{7}{2}}\pi^{\frac{1}{2}}\alpha^{\frac{5}{2}}(\sin\theta)^{\frac{1}{2}}}\left[\frac{e^{i\alpha\sin\theta}}{(1+\sin\theta)^{\frac{1}{2}}}+i\frac{e^{-i\alpha\sin\theta}}{(1-\sin\theta)^{\frac{1}{2}}}\right]-$$
$$-\frac{e^{-2i\alpha}}{2^{\frac{7}{2}}\pi\alpha^3(\sin\theta)^{\frac{1}{2}}}\left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{i\alpha\sin\theta}}{1-\sin\theta}+i\left(\frac{1-\sin\theta}{2}\right)^{\frac{1}{2}}\frac{e^{-i\alpha\sin\theta}}{1+\sin\theta}\right]+$$
$$+O(\alpha^{-\frac{7}{2}}). \qquad (65)$$

Similarly, the second carder contribution is given by

$$- \int_1^{\infty} \psi_0(w) J_0(\alpha w \sin\theta dw -$$

$$\int_1^{1+i\infty} \left(\frac{w}{1-w}\right)^{\frac{1}{2}} \psi_3(-w) e^{i\alpha w} J_0(\alpha w \sin\theta) dw \sim$$

$$\sim \frac{(2^{\frac{1}{2}} - 1) e^{-2i\alpha}}{16 \pi\alpha^3 (\sin\theta)^{\frac{1}{2}}} \left[\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{i\alpha\sin\theta}}{1-\sin\theta} + i\left(\frac{1+\sin\theta}{2}\right)^{\frac{1}{2}} \frac{e^{-i\alpha\sin\theta}}{1+\sin\theta} \right]$$

$$+ O(\alpha^{-\frac{7}{2}}). \qquad (66)$$

Subsequent orders may be neglected if the degree of accuracy is to $O(\alpha^{-\frac{7}{2}})$.

10. AN EXPLICIT FORM OF THE FAR SCATTERED FIELD.

An integral form of the scattered field, $u_s(\underset{\sim}{R})$, is given in (40). It has been shown that this can be written,

$$\underset{s}{u}(\underset{\sim}{R}) \sim -\frac{1}{2}\frac{e^{-i\alpha R}}{R}F(\theta), \tag{67}$$

where F(θ) is given by (48). The zero, first and second order contributions to F(θ) are given by (56), (65) and (66), respectively. Combining these expressions shows that,

$$
\begin{aligned}
F(\theta) \sim {} & \left(\frac{2}{\pi\alpha}\right)^{\frac{1}{2}} \frac{e^{-i\frac{\pi}{4}}}{(\sin\theta)^{\frac{3}{2}}} \Bigg\{ \Bigg[(1+\sin\theta)^{\frac{1}{2}} e^{i\alpha\sin\theta} - i(1-\sin\theta)^{\frac{1}{2}} e^{-i\alpha\sin\theta} \Bigg] \\
& + \frac{i}{\alpha\sin\theta} \Bigg(\frac{5}{8} \Bigg[(1+\sin\theta)^{\frac{1}{2}} e^{i\alpha\sin\theta} + i(1-\sin\theta)^{\frac{1}{2}} e^{-i\alpha\sin\theta} \Bigg] - \\
& \qquad - \frac{1}{4} \Bigg[(1+\sin\theta)^{\frac{3}{2}} e^{i\alpha\sin\theta} + i(1-\sin\theta)^{\frac{3}{2}} e^{-i\alpha\sin\theta} \Bigg] \Bigg) - \\
& - \frac{1}{(\alpha\sin\theta)^2} \Bigg(\frac{1}{16} \Bigg[(1+\sin\theta)^{-\frac{1}{2}} e^{i\alpha\sin\theta} - i(1-\sin\theta)^{-\frac{1}{2}} e^{-i\alpha\sin\theta} \Bigg] - \\
& \qquad - \frac{55}{128} \Bigg[(1+\sin\theta)^{\frac{1}{2}} e^{i\alpha\sin\theta} - i(1-\sin\theta)^{\frac{1}{2}} e^{-i\alpha\sin\theta} \Bigg] + \\
& \qquad + \frac{11}{32} \Bigg[(1+\sin\theta)^{-\frac{3}{2}} e^{i\alpha\sin\theta} - i(1-\sin\theta)^{\frac{1}{2}} e^{-i\alpha\sin\theta} \Bigg] - \\
& \qquad - \frac{3}{32} \Bigg[(1+\sin\theta)^{-\frac{5}{2}} e^{i\alpha\sin\theta} - i(1-\sin\theta)^{\frac{5}{2}} e^{-i\alpha\sin\theta} \Bigg] \Bigg) \Bigg\} + \\
& + \frac{ie^{-2i\alpha}}{2^{\frac{5}{2}}\pi\alpha^{2}(\sin\theta)^{\frac{1}{2}}} \Bigg\{ \Bigg[\frac{(1+\sin\theta)^{\frac{1}{2}}}{1-\sin\theta} e^{i\alpha\sin\theta} + \frac{(1-\sin\theta)^{\frac{1}{2}}}{1+\sin\theta} e^{-i\alpha\sin\theta} \Bigg] + \\
& \quad + \frac{i}{\alpha} \Bigg(\frac{9}{8} \Bigg[\frac{(1+\sin\theta)^{\frac{1}{2}}}{(1-\sin\theta)^{2}} e^{i\alpha\sin\theta} + i\frac{(1-\sin\theta)^{\frac{1}{2}}}{(1+\sin\theta)^{2}} e^{i\alpha\sin\theta} \Bigg] - \\
& \qquad - \frac{3}{16} \Bigg[\frac{(1+\sin\theta)^{\frac{3}{2}}}{(1-\sin\theta)^{2}} e^{i\alpha\sin\theta} + i\frac{(1-\sin\theta)^{\frac{3}{2}}}{(1+\sin\theta)^{2}} e^{i\alpha\sin\theta} \Bigg] + \\
& \qquad + \frac{1}{8\sin\theta} \Bigg[\frac{(1+\sin\theta)^{\frac{1}{2}}}{1-\sin\theta} e^{i\alpha\sin\theta} + i\frac{(1-\sin\theta)^{\frac{1}{2}}}{1+\sin\theta} e^{-i\alpha\sin\theta} \Bigg] \Bigg) \Bigg\} - \\
& - \frac{e^{-2i\alpha}}{2^{\frac{7}{2}}\pi\alpha^{3}(\sin\theta)^{\frac{1}{2}}} \Bigg[\frac{(1+\sin\theta)^{\frac{1}{2}}}{(1-\sin\theta)} e^{i\alpha\sin\theta} + i\frac{(1-\sin\theta)^{\frac{1}{2}}}{(1+\sin\theta)} e^{-i\alpha\sin\theta} \Bigg] + \\
& + O(\alpha^{-\frac{7}{2}}) .
\end{aligned}
\tag{68}
$$

Equations (67) and (68) together give an expression for the scattered field which is independent or ϕ and is subject to the restrictions that $R \gg 1$ and $\phi \neq 0$ or $\frac{\pi}{2}$. These are the conditions for a far field off the axis of symmetry and away from the plane of the disc.

Keller [7], and Karp and Keller [6], have given the leading terms of both the singly diffracted field and the doubly diffracted field. The leading term of the singly diffracted field agrees with the first term of the zero order field in the present work, after a change of notation. The leading term of the doubly diffracted field agrees with the first term of the first order field. The results of Keller's approximate theory therefore agree with those of the present work, which are, however, more extensive and contain terms not included in Keller's theory.
It should be noted that Jones' iteration scheme gives iterates which correspond to fields produced by multiple diffractions. The k^{th} iterate corresponds to the field produced after k diffractions across the disc, the zero order field coming directly from a single diffraction of the incident field.

The author would like to thank Professor D.S.Jones for many enlightening discussions during the period in which the work presented in this report was prepared.

APPENDIX

We here consider the integral,

$$2\int_0^1 \frac{(1-t)^{\frac{1}{2}}\ t^{\frac{1}{2}}}{(t+\upsilon)(x-t)^2}\ dt\ , \tag{A1}$$

where x is a point on the negative imaginary axis.
On taking a Fourier transform with respect to v we have,

$$\int_{-\infty}^{\infty} e^{-i\beta\upsilon}\ 2\int_0^1 \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(t+\upsilon)(x-t)^2}\ dt\, d\upsilon =$$

$$= -2\pi i\ \mathrm{sgn}\,\beta \int_0^1 \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2}\ e^{i\beta t}\ dt \tag{A2}$$

Take branch lines from 1 to + ∞ for $(1 - t)^{\frac{1}{2}}$ and from $-\infty$ to 0 for $t^{\frac{1}{2}}$. If $\beta > 0$ we extend the contour of integration above the branch lines. The right hand side of (A2) may then be written

$$-2\pi i \left\{ \int_{-\infty}^{\infty} \frac{(1-t)^{\frac{1}{2}}\ t^{\frac{1}{2}}}{(x-t)^2}\ e^{i\beta t} dt - \int_{-\infty}^{0} \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t}\ dt\ - \right.$$

$$\left. - \int_1^{\infty} \frac{(1-t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t}\ dt \right\}\ =$$

$$= -2\pi \int_0^{\infty} \frac{(1+t)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x+t)^2} e^{i\beta t}\ dt\ +\ 2\pi \int_1^{\infty} \frac{(t-1)^{\frac{1}{2}} t^{\frac{1}{2}}}{(x-t)^2} e^{i\beta t}\ dt\ , \tag{A3}$$

after the first integral is deformed upwards to $+i\infty$ without contribution. If $\beta < 0$ we extend the contour below the "branch lines- The right hand side of (A2) is then

$$-2\pi i\left\{\int_{-\infty}^{\infty}\frac{(1-t)^{\frac{1}{2}}\,t^{\frac{1}{2}}}{(x-t)^2}\,e^{i\beta t}dt - \int_{-\infty}^{0}\frac{(1-t)^{\frac{1}{2}}\,t^{\frac{1}{2}}}{(x-t)^2}e^{i\beta t}\,dt - \right.$$

$$\left. -\int_{1}^{\infty}\frac{(1-t)^{\frac{1}{2}}\,t^{\frac{1}{2}}}{(x-t)^2}e^{i\beta t}\,dt\right\}.$$

The first contour may be deformed downwards to - i ∞. In doing so it loops round the singularity at t = x on the negative imaginary axis. Hence we obtain

$$-2\pi i\oint\frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{(x-t)^2}e^{i\beta t}\,dt \;-2\pi\int_0^{\infty}\frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{(x-t)^2}e^{-i\beta t}\,dt \;+$$

$$+2\pi\int_1^{\infty}\frac{(t-1)^{\frac{1}{2}}t^{\frac{1}{2}}}{(x-t)^2}e^{i\beta t}\,dt\,. \qquad \text{(A4)}$$

Combining (A3) and (AA-) we have, for all β,

$$\int_{-\infty}^{\infty}e^{-i\beta v}\,2\int_0^1\frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{(t+\upsilon)(x-t)^2}\,dt\,d\upsilon \;=$$

$$=2\pi\int_1^{\infty}\frac{(t-1)^{\frac{1}{2}}t^{\frac{1}{2}}}{(x-t)^2}\,e^{i\beta t}\,d_t \;-\; 2\pi\int_0^{\infty}\frac{(1+t)^{\frac{1}{2}}t^{\frac{1}{2}}}{(x-t)^2}\,e^{-i\beta t}\,d_t\;+$$

$$+\,H(-\beta)\,2\pi i\oint\frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{(x-t)^2}\,e^{i\beta t}\,dt\,. \qquad \text{(A5)}$$

Taking an inverse Fourier transform gives back the original integral (A1).Hence

$$2\int_0^1\frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{(t+\upsilon)(x-t)^2}\,dt \;=$$

$$=\,2\pi\frac{(1+\upsilon)^{\frac{1}{2}}\upsilon^{\frac{1}{2}}}{(x+u)^2}\{H(\upsilon)+H(-1-\upsilon)\}\;+$$

$$+\oint\frac{(1-t)^{\frac{1}{2}}t^{\frac{1}{2}}}{(t+\upsilon)(x-t)^2}\,dt\,. \qquad \text{(A6)}$$

REFERENCES

[1] JONES,D.S. Diffraction at high frequencies by a circular disc. Proc.Camb.Phil.Soc. 61 (1965), 223-245.

[2] JONES,D.S. Diffraction of short wavelengths by a rigid circular disc. Quart. J. Mech.Appl. Math. 18 (1965), 191 - 208.

[3] JONES,D.S. Diffraction of a high frequency electromagnetic wave by a perfectly conducting circular disc. Proc.Camb.Phil.Soc. 61 (1965), 247 -270,

[4] JONES,D.S. On a certain singular integral equation I. J.Math.Phys. 43 (1964), 27-33.

[5] JONES,D.S. On a certain singular integral equation II J.Math.Phys. 43(1964), 263-273.

[6] KARP, S.N. and KELLER, J.B. Multiple diffraction by an aperture in a hard screen. Optica Acta 8 (1961), 61-71.

[7] KELLER,J.B. Diffraction by an aperture I. J.Appl.Phys. 28 (1957), 426 - 444.

[8] KELLER,J.B. Errata: Diffraction by an aperture. J.Appl.Phys. 29 (1958), 744.

[9] KELLER,J.B., LEWIS,R.M. and SECKLER,B.D. Diffraction by an aperture II J.Appl.Phys. 28 (1957), 570- 579.

[10] LEVINE,H. Diffraction by a circular aperture at high frequencies. (N.Y.U. Res.Rep.EM-84(1955)).

[11] LEVTNE,H and WU,T.T. Diffraction by an aperture at high frequencies. (Stanford University Technical Report 71 (1957)).

[12] WATSON,G. N. A treatise on the theory of Bessel functions. (C.U.P., 2nd Ed., (1944)).