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On the numerical performance of a domain decomposition method for conformal mapping.

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1. Introduction

This paper is a sequel to a recent paper [14], concerning a domain decomposition method (hereafter referred to as *DDM*) for the conformal mapping of a certain class of quadrilaterals. For the description of the *DDM* we proceed exactly as in [14:§1], by introducing the following terminology and notations.

Let G be a simply-connected Jordan domain in the complex z - plane $(z = x + iy)$, and consider a system consisting of G and four distinct points z_1, z_2, z_3, z_4 in counter-clockwise order on its boundary ∂G. Such a system is said to be a quadrilateral Q and is denoted by $Q = \{G ; z_1, z_2, z_3, z_4 \}$. The conformal module $m(Q)$ of Q is defined as follows:

Let R be a rectangle of the form

$$R := \{ (\xi,\eta) : a < \xi < b, c < \eta < d \} , \qquad (1.1)$$

in the w-plane $(w = \xi + i\eta)$, and let h denote its aspect ratio, i.e. $h := (d-c)/(b-a)$. Then, $m(Q)$ is the unique value of h for which Q is conformally equivalent to a rectangle of the form (1.1), in the sense that for $h = m(Q)$ and for this value only there exists a unique conformal map $R \rightarrow G$ which takes the four comers $a + ic$, $b + ic$, $b + id$, and $a + id$, of R respectively onto the four points z_1, z_2, z_3, z_4. In particular, $h = m(Q)$ is the only value of h for which Q is conformally equivalent to a rectangle of the form

$$R_h\{a\} := \{ (\xi,\eta) : 0 < \xi < 1, \alpha < \eta < \alpha + h \} . \qquad (1.2)$$

The *DDM* is a method for computing approximations to the conformal modules and associated conformal maps of quadrilaterals of the form illustrated in Figure 1.1 (b). That is, the method is concerned with the mapping of quadilaterals

$$Q := \{ G ; z_1, z_2, z_3, z_4 \}, \qquad (1.3a)$$

where:

- The domain G is bounded by the straight lines x = 0 and x = 1 and two Jordan arcs with cartesian equations $y = \tau_1(x)$ and $y = \tau_2(x)$, where τ_j ; $j = 1, 2$, are positive in [0,1], i.e.

$$G := \{ (x, y): 0 < x < 1, -\tau_1(x) < y < \tau_2(x) \}. \tag{1.3b}$$

- The points z_1, z_2, z_3, z_4 are the corners where the arcs intersect the straight lines, i.e.

$$z_1 = -i\tau_1(0),\ z_2 = 1 - i\tau_1(1),\ z_3 = 1 + i\tau_2(1),\ z_4 = i\tau_2(0). \tag{1.3c}$$

Let Q be of the form (1.3) and let

$$G_1 := \{ (x, y): 0 < x < 1, -\tau_1(x) < y < 0 \}, \tag{1.4a}$$

and

$$G_2 := \{ (x, y): 0 < x < 1, 0 < y < \tau_2(x) \}, \tag{1.4b}$$

so that $\overline{G} = \overline{G_1} \cup \overline{G_2}$. Also, let Q_1 and Q_2 denote the quadrilaterals

$$Q_1 := \{G_1; z_1, z_2, 1, 0\} \text{ and } Q_2 := \{G_2; 0, 1, z_3, z_4\}, \tag{1.4c}$$

and let $h := m(Q)$ and $h_j := m(Q_j)$; $j = 1,2$; see Figures 1.2(b) and 1.3(b). Finally, let g and g_j ; $j = 1,2$, denote the conformal maps

$$g := R_h\{-h_1\} \rightarrow G, \tag{1.5}$$

$$g_1 : R_{h_1}\{-h_1\} \rightarrow G_1 \text{ and } g_2 : R_{h_2}\{0\} \rightarrow G_2, \tag{1.6}$$

where, with the notation (1.2),

$$R_h\{-h_1\} := \{ (\xi, \eta): 0 < \xi < 1, -h_1 < \eta < h - h_1 \},$$

$$R_{h_1}\{-h_1\} := \{ (\xi, \eta): 0 < \xi < 1, -h_1 < \eta < 0 \},$$

$$R_{h_2}\{0\} := \{ (\xi, \eta): 0 < \xi < 1, 0 < \eta < h_2 \};$$

see Figures 1.1-1.3. Then, the *DDM* consists of the following:

(*a*) Subdividing the quadrilateral Q, given by (1.3), into the two smaller quadrilaterals Q_1 and Q_2, given by (1.4).

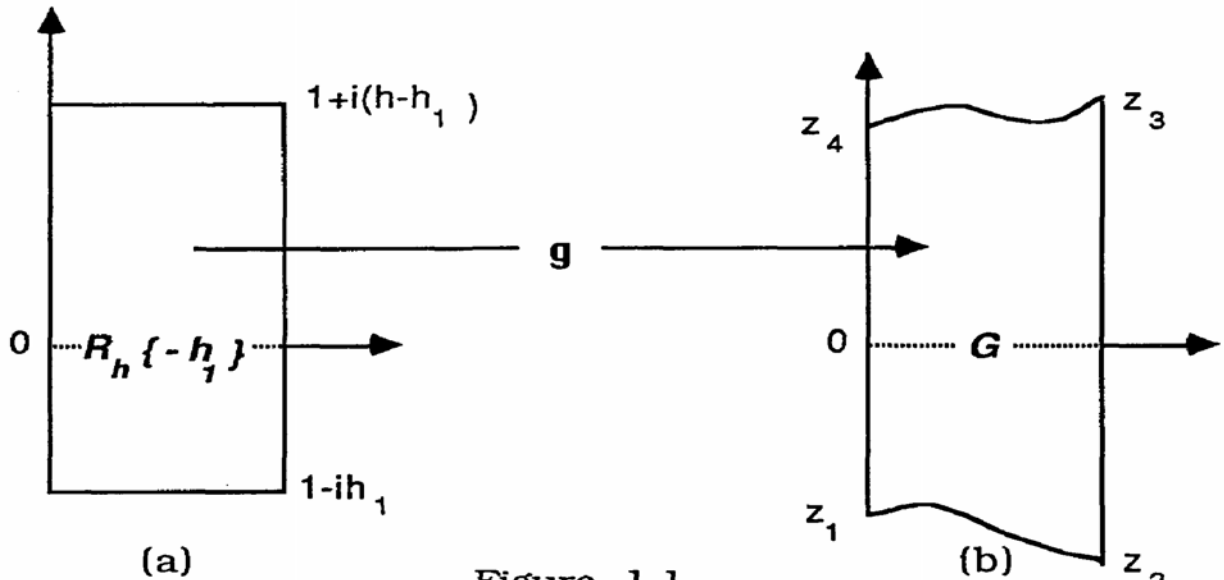

Figure 1.1

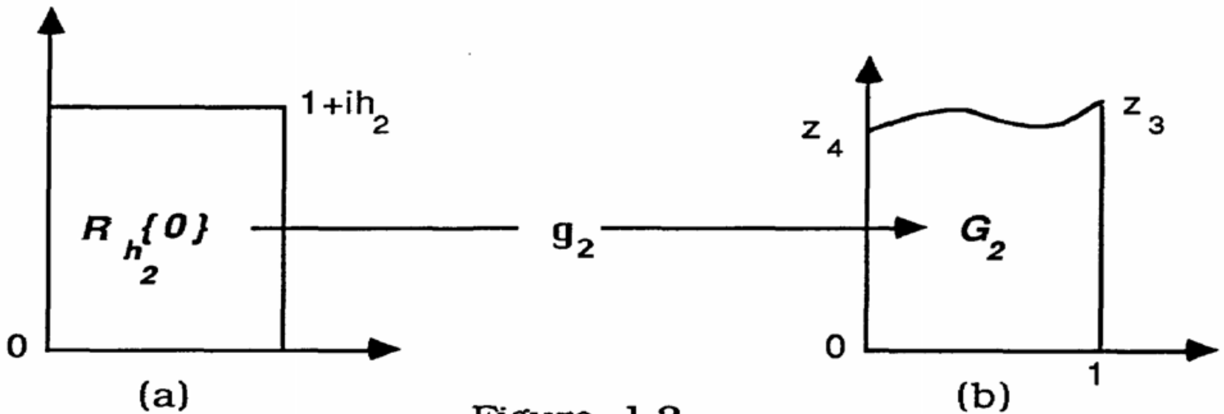

Figure 1.2

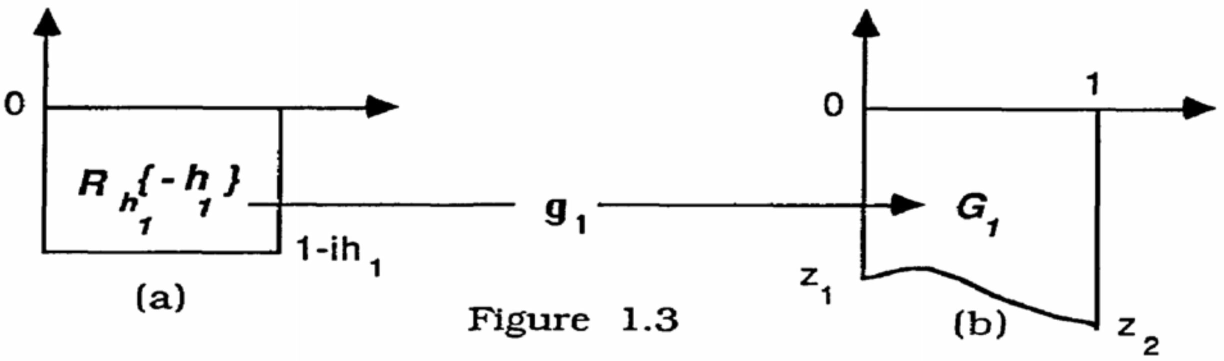

Figure 1.3

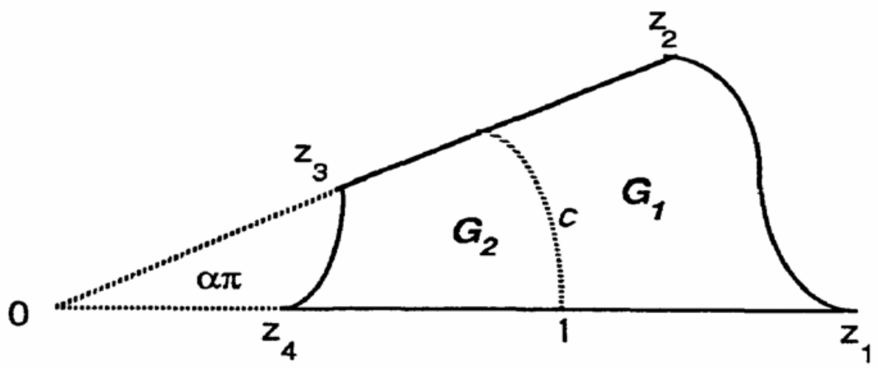

Figure 1.4

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(b) Approximating the conformal module of Q by the sum of the conformal modules of Q_1 and Q_2, i.e. approximating h by

$$\bar{h} := h_1 + h_2 . \tag{1.7a}$$

(c) Approximating the rectangle $R_h\{-h_1\}$ and the conformal map $g : R_h\{-h_1\} \to G$ respectively by

$$R_{\bar{h}}\{-h_1\} := \{(\xi,\eta) : 0 < \xi < 1, -h_1 < \eta < h_2\} , \tag{1.7b}$$

and

$$\bar{g}(w) = \begin{cases} g_2(w) R_{h_2}\{0\} \to G_2, & \text{for } w \in R_{h_2}\{0\}, \\ g_1(w) R_{h_1}\{-h_1\} \to G_1, & \text{for } w \in R_{h_1}\{-h_1\}. \end{cases} \tag{1.7c}$$

The initial motivation for considering the above method came from : (a) The intuitive observation that if the constituent quadrilterals Q_1 and Q_2 are both "long" then $\bar{h}$ is close to h. (b) Experimental evidence indicating that $\bar{h}$ is close to h even when Q_1 and Q_2 are only moderately long; see [12:§5] and [14:§1]. (It is important to note that $h \geq h_1 + h_2$ and equality occurs only in the two trivial cases where G is a rectangle or $\tau_1(x) = \tau_2(x)$, $x \in [0,1]$; see e.g. [9:p.437].)

The treatment of the *DDM* contained in [14] is a theoretical investigation leading to estimates of the errors in the approximations (1.7). These error estimates are derived by assuming that the functions τ_j ; $j = 1,2$, satisfy the following:

Assumptions A1. 1

(i) τ_j ; j = 1,2, are absolutely continuous in [0,1], and

$$d_j := \operatorname*{ess\ sup}_{0 \leq x \leq 1} |\tau'_j(x)| < \infty . \tag{1.8}$$

(ii) If

$$m_j := \max_{0 \leq x \leq 1} \{ \exp(-\pi \tau_j(x)) \} ; j = 1,2, \tag{1.9a}$$

then

$$\varepsilon_j := d_j \{((1 + m_j)/(1 - m_j)\} < 1; \ j = 1,2. \square \tag{1.9b}$$

In addition to the theory, [14] also contains two numerical examples comparing the actual errors in the *DDM* approximations with those predicted by the theoretical estimates.

The present paper is concerned with the numerical performance of the *DDM* and, in particular, with the performance of the method in cases where the functions τ_j ; $j = 1,2$, do not fulfil the rather restrictive condition (1.9) needed for the theory of [14]. More specifically, the main purpose of this paper is to show by means of numerical examples that some of the theoretical results of [14] remain valid even when the condition (1.9) is violated, and thus to provide experimental support for certain conjectures made in [14].

We end this introductory section by making the following remarks concerning the *DDM* and related matters:

- A survey of available methods for computing approximations to the conformal modules and the associated conformal maps of general quadrilaterals is given in [12], where also several areas of application of the conformal maps are discussed; see also [4] - [7] and [9:§16.11]. □

- Although this is not considered here, the *DDM* can also be applied to quadrilaterals of the form illustrated in Figure 1.4, provided that the crosscut c of subdivision is taken to be a circular arc; see [14:Remark 4.7]. In other words, the application of the *DDM* is restricted to quadrilaterals that have one of the two special forms illustrated in Figures 1.1 and 1.4. We note however that the mapping of such quadrilaterals has received considerable attention recently; see e.g. [2], [8], [11], [12], [15] and [17]. □

- A more general form of the *DDM* involves subdividing the original quadrilateral Q into two quadrilaterals Q_1 and Q_2 of the form (1.4) at the lower and upper ends, and a rectangle in the middle. This can be described more precisely as follows:

Let

$$G := \{ (x,y) : 0 < x < 1, -\tau_1(x) < y < \tau_2(x) + c \} ,$$

where c > 0, let

$$G_1 := \{ (x,y) : 0 < x < 1, -\tau_1(x) < y < 0 \} ,$$

and

$$G_2 := \{ (x,y) : 0 < x < 1, c < y < \tau_2(x) + c \},$$

so that $\overline{G} = \underline{\overline{G_1} \cup \overline{R_c} \{0\} \cup \overline{G_2}}$, and let

$$z_1 = -i\tau_1(0), z_2 = 1 - i\tau_1(1), z_3 = 1 + i(\tau_2(1) + c), \ z_4 = i(\tau_2(0) + c).$$

Then the general form of the *DDM* consists of the following:

(a) Subdividing the quadrilateral $Q := \{G; z_1, z_2, z_3, z_4\}$ into three smaller quadrilaterals, i.e. the quadrilaterals $Q_1 := \{G_1; z_1, z_2, 1, 0\}$ and $Q_2 := \{G_2; ic, 1+ic, z_3, z_4\}$, at the lower and top ends, and the rectangular quadrilateral $\{R_c\{0\}; 0, 1, 1+ic, ic\}$, in the middle.

(b) Approximating the conformal module $h := m(Q)$ by

$$\bar{h} := h_1 + h_2 + c, \tag{1.10}$$

where $hj := m(Qj)$; $j = 1,2$.

(c) Approximating the rectangle $R_h\{-h_1\}$ and the conformal map $g : R_h\{-h_1\} \to G$ respectively by $R_{\bar{h}}\{-h_1\}$ and

$$\bar{g}(w) := \begin{cases} g_2(w) : R_{h_2}\{c\} \to G_2, & \text{for } w \in R_{h_2}\{c\}, \\ w, & \text{for } w \in R_c\{0\} \\ g_1(w) : R_{h_1}\{-h_1\} \to G_1, & \text{for } w \in R_h\{-h_1\}. \end{cases} \quad , \square \tag{1.11}$$

- The *DDM* is of practical interest for the following reasons:

(i) Given the conformal modules and associated conformals maps of two quadrilaterals Q_1 and Q_2 of the form (1.4), the method provides approximations to the conformal module and associated conformal map of any quadrilateral consisting of $Q_1$and Q_2, at the lower and top ends, and a rectangle of any height in the middle.

(ii) The method can be used to overcome the “crowding“ difficulties associated with the numerical conformal mapping of “long” quadrilaterals of the form (1.3). (Full details of the crowding phenomenon and its damaging effects on numerical procedures for the mapping of “long”

quadrilaterals can be found in [12], [13] and [10]; see also [4:p.179], [9:p.428] and [16 :p.4].)

(iii) Numerical methods for approximating the conformal maps of quadrilaterals of the form (1.4) are often substantially simpler than those for quadrilaterals of the more general form (1.3); see e.g. [8], and [12:§3.4]. □

- Of the two conditions involved in the assumptions A 1.1, only Al. l(ii) is restrictive from the practical point of view. This condition is more or less equivalent to requiring that the slopes of the two curves $y = \tau_j(x)$; $j = 1,2$, are numerically less than unity in [0,1]. This is so, because the values m_j ; j = 1,2, given by (1.9a) are "small", even when the quadrilaterals $Q_{j,}$; j = 12, are only moderately "long". □

2. Theoretical error estimates

As in Section 1, let Q and Q_j : j = 1,2, denote the three quadrilaterals defined by (1.3) and (1.4), let $h := m(Q)$ $h_j := m(Q_j)$; j = 1,2, and let g, g_j, j = 1,2, be the associated conformal maps (1.5) and (1.6). Also, let

$$\mathrm{x}(\xi) := \mathrm{Re}\, g(\xi - ih_1), \quad \hat{\mathrm{x}}(\xi) := \mathrm{Re}\, g\ (\xi + i(h - h_1)),$$

and

$$\mathrm{x}_1(\xi) := \mathrm{Re}\, g_1(\xi - ih_1), \quad \hat{\mathrm{x}}_2(\xi) := \mathrm{Re}\, g_2(\xi + ih_2)\ ,$$

and let E_h and $E_g\{j\}$, $E_X\{j\}$, $E_\tau\{j\}$; $j = 1,2$, denote the following domain decomposition errors:

$$E_h := h - (h_1 + h_2)\ , \tag{2.1}$$

$$E_g\{1\} := \max\{\, |\, g(w) - g_1(w)\, | : w \in \overline{R_{h_1}}\ \{-h_1\}\ \}\ , \tag{2.2a}$$

$$E_g\{2\} := \max\ \{\, |\ g(w + i\,E_h) - g_2(w)\ |\ : w \in \overline{R_{h_1}}\{0\}\ \}\ , \tag{2.2b}$$

$$E_\mathrm{x}\{1\} := \max_{0 \le \xi \le 1} |\mathrm{x}(\xi) - \mathrm{x}_1(\xi)|. \quad E\mathrm{x}\{2\} := \max_{0 \le \xi \le 1} \left|\hat{\mathrm{x}}(\xi) - \hat{\mathrm{x}}_2(\xi)\right|, \tag{2.3}$$

and

$$E_\tau\{1\} := \max_{0 \le \xi \le 1} \left|\tau_1(X(\xi)) - \tau_1(X_1(\xi))\right|,\ E_\tau\{2\} := \max_{0 \le \xi \le 1} \left|\tau_2(X(\xi)) - \tau_2(X_2(\xi))\right|. \tag{2.4}$$

Finally, assume that the functions τ_j ; j =1,2, satisfy the assumptions A1. 1, and let

$$\alpha(\varepsilon_j,\varepsilon) := 2/\{(1-\varepsilon_j)(1-\varepsilon^2)\}^{1/2} \ ; j = 1,2, \tag{2.5a}$$

and

$$\beta(\varepsilon_j,\ \varepsilon) := \sqrt{8}\ /\{(1-\varepsilon_j)\ (1-\varepsilon^2)\}^{1/2}; \quad j = 1,2, \tag{2.5b}$$

where the ε_j ; j = 1,2, are given by (1.9), and $\varepsilon := \max(\varepsilon_1, \varepsilon_2)$. Then, the main results of [14] are the following estimates of the errors E_h and $E_g\ \{j\}$; j = 1,2, in the *DDM* approximations (1.7):

$$E_h \le \pi^{-1}\ d_1\,\alpha(\varepsilon_1,\varepsilon)\ \{\varepsilon_1\ e^{-2\pi\pi_1} + \varepsilon_2 e^{-\pi h}\} + \pi^{-1} d_2 \alpha(\varepsilon_2,\varepsilon)\ \{\varepsilon_2 e^{-2\pi\pi_2} + \varepsilon_1 e^{-\pi h}\}, \tag{2.6}$$

and

$$E_g\ \{j\} \le \max\ \{M_j, N_j\};\ j = 1,2, \tag{2.7a}$$

where

$$M_j := \pi^{-1/2}(1+d_j^2)^{1/2}\,\beta(\varepsilon_j,\varepsilon)\ \{\varepsilon_j + \varepsilon_{3-j} e^{-\pi h}\}^{1/2}\ \{\varepsilon_j e^{-2\pi h_j} + \varepsilon_{3-j} e^{-\pi h}\}^{1/2}, \tag{2.7b}$$

$$N_j := \frac{1}{2}\,\pi^{-1/2}\,\beta(0,\varepsilon)\,\{5\varepsilon_j e^{-\pi h_j} + 3\varepsilon_{3-j} e^{-\pi(h-h_j)}\} + \pi^{-1}\ d_j\ \alpha(\varepsilon_j,\varepsilon)\ \{\varepsilon_j\ e^{-2\pi\pi_j} + \varepsilon_{3-j} e^{-\pi h}\}; \tag{2.7c}$$

see [14:Thms 4.1, 4.4]. Since $h \ge h_1 + h_2$, the above estimates show that if the functions τ_j ; j= 1,2, satisfy the assumptions Al. l, then

$$E_h = O\{exp\ (-2\pi\pi^*)\}, \tag{2.8}$$

and

$$E_g\ \{j\} = O\{exp(-\pi h^*)\}; \quad j = 1,2, \tag{2.9}$$

where $h^* := \min(h_1, h_2)$.

In addition to (2.6) and (2.7), [14] also contains estimates of the errors $E_X\{j\}$ and $E_\tau\{j\}$; j = 1,2; see Theorem 4.2 and Remark 4.2 in [14]. These estimates show that under the assumptions A1.1,

$$E_X\{j\} = O\{\, exp(-\pi h^*)\} \quad \text{and} \quad E_\tau\{j\} = O\{\, exp(-\pi h^*)\}\,; \quad j = 1,2, \tag{2.10}$$

where as before $h^* := \min(h_1, h_2)$. Finally, [14] contains a theorem (Theorem 4.3), which shows that if $Q_1$and Q_2 are both "long " quadrilaterals, then at points sufficiently far from the two sides $\eta = -h_1$ and $\eta = h - h_1$ of $R_h\{-h_1\}$, the conformal map g can be approximated closely by the identity map. In particular, this theorem of [14] shows that

$$\max_{0 \le \xi \le 1} \left| g(\xi + i0) - \xi \right| = O\{\, exp(-\pi h^*)\,\} \tag{2.11}$$

The method of analysis used hi [14] for deriving the above results makes extensive use of the theory given in [3: Kap.V, §3], in connection with the integral equation method of Garrick, for the conformal mapping of doubly-connected domains. This involves expressing the three problems for the conformal maps g and g_j ; $j = 1,2$, as equivalent problems for the conformal maps of three symmetric doubly-connected domains; see [8], [12:§3.2,3.4] and [13:§3]. (With reference to the above comment, readers who are familiar with the method of Garrick will recognize the very close similarity between the condition (1.9), which is needed for the analysis used in [14], and the so-called $\varepsilon\delta$ - condition needed for the theory of the Garrick method.)

We end this section by observing that the results of [14] simplify considerably in the case where one of the two subdomains G_1 or G_2 is a rectangle. For example, let $\tau_1(x) = c > 0$, $x \in [0, 1]$, i.e. let

$$G_1 := \{(x,y) : 0 < x < 1, -c < y < 0\} = R_c\{-c\}\ . \tag{2.12}$$

Then, $g_1(w) = w$, $h_1 = c$, $d_1 = \varepsilon_1 = 0$ and, for any value $c > 0$, the results (2.8)-(2.10) simplify as follows:

$$E_h := h - (c + h_2) = O\{exp(-2\pi\pi_2)\}\ , \tag{2.13}$$

$$E_g\{j\} = O\{exp(-\pi h_2)\}\,; \quad j = 1,2, \tag{2.14}$$

$$E_X\{\,j\,\} = O\{\, exp(-\pi h_2)\},\qquad E_\tau\{\,j\,\} = O\{\, exp(-\pi h_2)\};\quad j = 1,2. \tag{2.15}$$

Also, in place of (2.11) we now have that for any point $w := \xi + i\,\eta \in R_c\{-c\}$,

$$\left| g(w) - w \right| = O\{ \ exp \ (-\pi (h_2 - \eta)) \}. \qquad (2.16)$$

Furthermore, all the above simplified results hold under the less restrictive assumptions obtained by replacing the inequalities (1.9) of A1. 1 (ii) by

$$\varepsilon_2 := d_2 \{ (1 + m2^2) / (1 - m_2^{\ 2}) \} < 1 \ ; \qquad (2.17)$$

see [14:Remark4.5].

3. Numerical results and discussion

In addition to the theoretical estimates summarized in Section 2, [14:§5] contains two numerical examples in which the quadrilaterals are chosen so that the functions τ_j ; $j = 1,2$, satisfy the assumptions A1.1. The numerical results of these examples confirm the theory of [14], and indicate that the *DDM* is capable of producing approximations of high accuracy, even when the quadrilaterals under consideration are only moderately long.

In this section we study further the numerical performance of the *DDM*. but here we consider its application to quadrilaterals that do not satisfy the condition (1.9) of the assumptions A1.1. That is, we are concerned with cases for which the theory of [14] does not apply. Our main purpose is to provide experimental evidence supporting the following two conjectures made in [14]:

C3.1 : The results (2.8)-(2.9) hold even when the condition (1.9) is not fulfilled; see [14:Remark 5.4]. More specifically, the claim here is that

$$E_h = O \ \{ \ exp \ (-2\pi h^* \ ) \ \}, \qquad (3.1)$$

and

$$E_g\{ \ j \ \} = O\{ \ exp \ (-\pi h^*) \ \} \ ; \quad j = 1,2, \qquad (3.2)$$

with $h^* := \min (h_1, h_2)$, even when $dj \geq 1$; $j = 1,2$, where d_j are the values given by (1.8). □

C3.2 : The errors E_X {j} and $E_\tau\{j\}$; $j = 1,2$, are $O\{ \ exp \ (-2\pi h^*) \}$, rather than $O \{ exp \ (-\pi h^* \ ) \}$ as predicted by the theory of [14]; see (2.10) and [14:Remark 5.2]. That is, the claim here is that

$$E_X\{\mathrm{j}\} = O\{\, exp(-2\pi h^*)\,\} \;\text{ and }\; E_\tau\{\, j\,\} = O\,\{\, exp(-2\pi h^*)\,\}\,;\;\; j=1{,}2, \qquad (3.3)$$

and that the above results hold even when the condition (1.9) is violated. □

Each of the three examples considered below involves the mapping of a quadrilateral Q of the form (1.3) and, in each case, the decomposition is performed by subdividing Q into two quadrilaterals Q_j ; j = 1,2, of the form (1.4). In presenting the numerical results we use the following notations:

- E_h and $E_g\,\{j\},\; E_X\,\{j\},\, E_\tau\,\{j\}$; $j=1{,}2$: As before these denote the actual *DDM* errors (2.1)-(2.4). More precisely, the values listed in the examples are reliable estimates of the errors (2.1)-(2.4). They are determined, as in [14:§5], from accurate approximations to h, h_j ; $j=1{,}2$, and g , g_j ; j = 1.2, which are computed by using the iterative algorithms of [8]. In particular, $E_g^{\{j\}}$; j = 1,2, are the maxima of two sets of values which are obtained by sampling respectively the approximations to the functions $g\,(w)$ - $g_1(w)$ and $g\,(w + iE_h)$ — $g_2\,(w)$ at a number of test points on the boundary segments $\eta = -h_1, 0$ of $R_{h_1}\{-h_1\}$ and $\eta = 0, h_2$ of $Rh_2\{\,0\,\}$. The values $E_X^{\{j\}}$ and $E_\tau^{\{j\}}$ are determined in a similar manner, by sampling the approximations to the functions $X(\xi)-X_1(\xi), \hat{X}(\xi)-\hat{X}_2(\xi)$, e.t.c. at a number of test points in $0 \le \xi \le 1$. (The only exception to the above are the values of E_h given in Example 3.1, in which Q is a trapezium and the subdivision consists of a smaller trapezium Q_2 and a rectangle $Q_1 := R_c\{-c\,\}$. In this case, $h := m(Q)$ and $h_2 := m(Q_2)$ are known exactly in terms of elliptic integrals. Hence $E_h := h - (c - h_2)$ is also known exactly.)

- δ_h and $\delta_g\,\{\,j\,\}, \delta_X\{\,j\,\}, \delta_\tau\,\{\mathrm{j}\}$; $j = 1, 2$: These denote the values used for testing the validity of (3.1)-(3.3). zhey are determined from the computed values of the errors E_h and $E_g\,\{\,j\,\}, E_X\{\,j\,\}, E_\tau\{\,j\,\}$ j = 1,2, as follows:

In each of the Examples 3.2 and 3.3, the functions τ_j ; j = 1,2, are of the form

$$\tau_j(x) := \sigma_j(x) + l\;;\quad j=1{,}2\,,$$

where $l \geq 0$, and in each case the values of the conformal modules and the errors in the *DDM* approximations are computed for several values of the parameter l. Let $h_1(l)$ and $h_2(l)$ denote the conformal modules of Q_1 and Q_2 corresponding to the value l, and let $h^*(l) := \min (h_1(l), h_2(l))$. Also, let E denote any of the errors E_h, $E_g\{j\}$, $E_X\{j\}$ or $E_\tau\{j\}$, and let $E(l)$ be the value of E corresponding to l. Then, the validity of (3.1) - (3.3) is checked by assuming that

$$E = O\{ \exp(-\delta\pi h^*) \} ; \quad h^* := \min(h_{1,} h_2),$$

and computing various values of δ (i.e. of δ_h, $\delta_g\{ j \}$, $\delta_X\{ j \}$ or $\delta_r\{ j \}$) by means of the formula

$$\delta = -\{ \log[E(l_1) / E(l_2)] / \{ \pi [h^*(l_1) - h^*(l_2)] \} .$$

(In the examples, l_1 and l_2 are taken to be successive values of the parameter l for which numerical results are listed.)

In the first example, i.e. in Example 3.1, we consider only the error E_h and, because of the form of Q, we check the validity of (2.13) rather than (3.1). That is we assume that

$$E_h = O\{ \exp(\delta_h \pi h_2) \} ,$$

and determine δ_h from the listed values of E_h, by modifying in an obvious manner the procedure described above.

Example 3.1 (See also [12:§5]) Q is the trapezium illustrated in Figure 3.1. That is, Q is defined by the functions

$$\tau_1(x) = c \qquad \text{and} \qquad \tau_2(x) = x - 1 + 1 \ ,$$

where $c > 0$ and $l > 0$. Here, $d_2 = 1$ and, because of this, the theory of [14] does not apply.

As was previously remarked, in this case the conformal modules $h := m(Q)$ and $h_2 := m(Q_2)$ are known in terms of elliptic integrals. Thus, Table 3.1 contains the exact values of h and h_2 corresponding to the parameters l = 1.25, 2.00, 2.50, 4.00, 5.00, and c = 0.75, 0.50, 1.50, 1.00, 5.00. (These were determined correct to twelve decimal places, by using the formulae of Bowman [1:p.l04].) The table also contains the values of the error $E_h := h - (h_2 + c)$ and, where possible, the corresponding values of δ_h. These values of δ_h indicate clearly that, for any c > 0, $E_h = O\{ \exp(-2\pi h_2) \}$. □

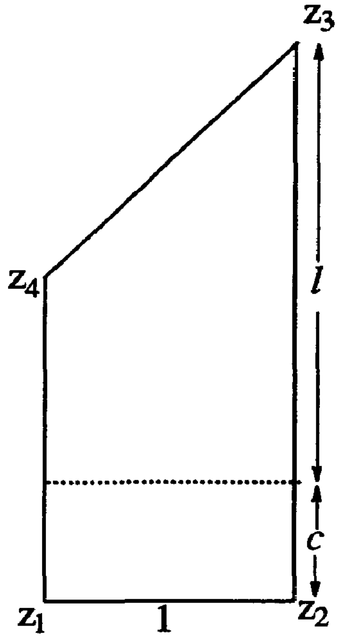

Figure 3.1

l	c	h_2	h	E_h	δ_h
1.25	0.75	0.516 810 878 029	1.279 261 571 171	1.2 E-02	-
2.00	0.50	1.279 261 571 171	1.779 359 959 478	9.8 E-05	2.021
2.50	1.50	1.779 359 959 478	3.279 364 399 489	4.4 E-06	1.972
4.00	1.00	3.279 364 399 489	4.279 364 399 847	3.6 E-10	1.999
5.00	5.00	4.279 364 399 847	9.279 364 399 847	-	-

Table 3.1

Example 3.2 Q and Q_j ; 1,2, are defined by (1.3) and (1.4) with

$$\tau_1(x) = 1 + 0.25\cos(2\pi x) + l \quad \text{and} \quad \tau_2(x) = 0.25x^4 - 0.5x^2 + 1 + l$$

Since $d_1 = \pi/2 > 1$, the above two functions do not satisfy the condition (1.9) needed for the theory of [14].

The numerical results corresponding to the values $l = 0.0(0.5)2.5$ are listed in Tables 3.2(a)-

3.2(c). Table 3.2(a) contains the computed values of the conformal modules, together with the estimates of $E_h := h - (h_1 + h_2)$ and the corresponding values of δ_h. (The values of h and h_j listed in the table are expected to be correct to all the figures quoted. The algorithms of [8] achieve this remarkable accuracy, because the two curves $y = \tau_j (x)$; $j = 1,2$, intersect the straight lines x = 0 and $x = 1$ at right angles; see [8:§6]) Tables 3.2(b) and (3.2c) contain respectively the estimates of the errors $E_g \{j\}$; $j = 1,2$, and $E_X \{j\}$, $E_\tau \{j\}$; $j = 1,2$, together with the values $\delta_g \{j\}$; $j = 1,2$, and

$\delta_X \{ j \}$, $\delta_\tau \{ j \}$; $j = 1,2$. □

l	h_1	h_2	h	E_h	δ_h
0.0	0.864 086 763 083	0.859 360 128 944	1.723 659 400 858	2.1 E-04	-
0.5	1.364 089 626 994	1.359 560 053 306	2.723 658 669 419	9.0 E-06	2.013
1.0	1.864 089 632 342	1.859 568 647 619	3.723 658 668 053	3.9 E-07	2.001
1.5	2.364 089 632 352	2.359 569 018 929	4.723 658 668 050	1.7 E-08	2.000
2.0	2.864 089 632 352	2.859 569 034 974	5.723 658 668 050	7.2 E-10	2.000
2.5	3.364 089 632 352	3.359 569 035 668	6.723 658 668 050	3.1 E-11	2.000

Table 3.2(a)

l	$E_g \{1\}$	$\delta_g \{1\}$	$E_g \{2\}$	$\delta_g \{2\}$
0.0	1.2 E-02	-	1.2 E-02	-
0.5	2.4 E-03	1.022	2.4 E.03	1.018
1.0	5.0 E-04	1.004	5.0 E-04	1003
1.5	1.0 E-04	1.000	1.0 E-04	1.000
2.0	2.1 E-05	1.000	2.1 E-05	1.000
2.5	4.5 E-05	1.000	4.5 E-06	1.000

Table 3.2(b)

l	$E_X\{1\}$	$\delta_X\{1\}$	$E_X\{2\}$	$\delta_X\{2\}$	$E_\tau\{1\}$	$\delta_\tau\{1\}$	$E_\tau\{2\}$	$\delta_\tau\{2\}$
0.0	4.9 E-04	-	1.1 E-03	-	5.2 E-04	-	4.1 E-04	-
0.5	2.1 E-05	1.999	4.6 E-05	2.001	2.2 E-05	2.005	1.7 E-05	2.002
1.0	9.1 E-07	2.000	2.0 E-06	2.000	9.6 E-07	2.000	7.6 E-07	2.000
1.5	3.9 E-08	2.000	8.6 E-08	2.000	4.2 E-08	2.000	3.3 E-08	2.000
2.0	1.7 E-09	2.000	3.7 E-09	2.000	1.8 E-09	2.000	1.4 E-09	2.000
2.5	7.3 E-11	2.000	1.6 E-10	2.000	7.8 E-11	2.000	6.1 E-11	2.000

Table 3.2(c)

Example 3.3 Q and Q_j ; $j = 1,2$, are defined by (1.3) and (1.4) with

$$\tau_1(x) = 0.75 + 0.2\,\mathrm{sech}^2(2.5x) + l \quad \text{and} \quad \tau_2(x) = x(1-x) + 1 + l\ .$$

In this case, the condition (1.9) is not fulfilled because $d_2 = 1$.

The numerical results corresponding to the values $l = 0.00\ (0.25)\ 1.25$ are listed in Tables 3.3(a)-3.3(c). (In this example, the values of h and h_j ; $j = 1,2$, listed in Table 3.3(a) are expected to be correct to eight significant figures.) ☐

l	h_1	h_2	h	E_h	δ_h
0.00	0.815 399 73	1.121 813 26	1.937 329 02	1.2 E-04	-
0.25	1.065 491 74	1.371 813 33	2.437 329 08	2.4 E-05	2.005
0.50	1.315 510 77	1.621 813 33	2.937 329 08	5.0 E-06	2.001
0.75	1.565 514 72	1.871 813 33	3.437 329 08	1.0 E-06	2.000
1.00	1.815 515 54	2.121 813 33	3.937 329 08	2.1 E-07	2.000
1.25	2.065 515 71	2.371 813 33	4.437 329 08	4.5 E-08	2.000

Table 3.3 (a)

l	$E_g\{1\}$	$\delta_g\{2\}$	$E_g\{2\}$	$\delta_g\{2\}$
0.00	8.9 E-03	-	8.8 E-03	-
0.25	4.0 E-03	1.027	3.9 E-03	1.017
0.50	1.8 E-03	1.011	1.8 E-03	1.007
0.75	8.1 E-04	1.005	8.1 E-04	1.003
1.00	3.7 E-04	1.002	3.7 E-04	1.001
1.25	1.7 E-04	1.000	1.7 E-04	1.000

Table 3.3 (b)

L	$E_X\{1\}$	$\delta_X\{1\}$	$E_X\{2\}$	$\delta_X\{2\}$	$E_\tau\{1\}$	$\delta_\tau\{1\}$	$E_\tau\{2\}$	$\delta_\tau\{2\}$
0.00	8.2 E-04	-	6.8 E -04	-	2.4 E-04	-	2.2 E-04	-
0.25	1.7 E-04	2.003	1.4 E-04	1.999	5.0 E-05	2.007	4.5 E-05	2.000
0.50	3.5 E-05	2.000	2.9 E-05	2.000	1.0 E-05	2.001	9.4 E-06	2.000
0.75	7.3 E-06	2.000	6.1 E-06	2.000	2.2 E-06	2.000	1.9 E-06	2.000
1.00	1.5 E-06	2.000	1.3 E-06	2.000	4.5 E-07	2.000	4.0 E-07	2.000
1.25	3.2 E-07	2.000	2.6 E-07	2.000	9.4 E-08	2.000	8.4 E-08	2.000

Table 3.3 (c)

The numerical results of Tables 3.1-3.3 indicate clearly that in the three examples considered above

$$\delta_h = 2, \quad \delta_g\{j\} = 1\,;\, j = 1,2, \quad \text{and} \quad \delta_X\{j\}, \delta_\tau\{j\} = 2\,;\; j = 1,2\,.$$

Thus, the numerical results provide experimental support for the conjectures C3.1 and C3.2, which were made in Remarks 5.2 and 5.4 of [14].

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