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SOME PROPERTIES OF CONTINUED FRACTIONS WITH APPLICATIONS IN MARKOV PROCESSES

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ABSTRACT

Several results for continued fractions are first derived and are then shown to be applicable to numerical solution of differential-difference equations arising from linear birth-death processes. These numerical solutions have a high degree of accuracy and the method gives rise to convergence when the birth-death process does not tend to a steady state.

1. Some Properties of Continued Fractions

We denote a continued fraction f_o by

$$f_o = \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \ldots\ldots + \frac{a_n}{b_n} \qquad (1.1)$$

where a_n and b_n are numbers, real or complex. The nth convergent of fis, $\frac{A_n}{B_n}$ where both A_n and B_n satisfy the recurrence relation

$$U_n = a_n U_n\text{-}2 + b_n U_n\text{-}1$$

with initial values $A_o = 0$, $A_1 = a_1$ and $B_0 = 1$, $B_1 = b_1$

Writing $\alpha_r = \prod_{i=l}^{r} a_i$ and using (l.2), the determinant formula

$$A_r B_{r+1} - A_{r+1} B_r = (-1)^r a_{r+1} \qquad (1.3)$$

may be obtained.

We now show that the set of recurrence relations

$$\left.\begin{array}{lllll} f_1 & = & a_1 & - & b_1 f_0 \\ f_2 & = & a_2 f_0 & - & b_2 f_1 \\ f_3 & = & a_3 f_1 & - & b_3 f_2 \\ & & \ldots\ldots & & \\ & & \ldots\ldots & & \\ f_{r+1} & = & a_{r+1} f_{r-1} & - & b_{r+1} f_r \end{array}\right\} \qquad (1.4)$$

$$\ldots\ldots$$

may be used to obtain the continued fraction (1.1). Dividing the first relation by f_o and rearranging, we have

$$f_0 = \frac{a_1}{b_1 + \frac{f_1}{f_0}} \qquad (1.5)$$

From the general relation, dividing by f_r , we have

$$\frac{f_r}{f_{r-1}} = \frac{a_{r+1}}{b_{r+1} + \frac{f_{r+1}}{f_r}} \qquad (16)$$

for r = 1,2,3, ... Results (1.5) and (1.6) lead to the continued fraction (1.1),for which we now establish an elementary convergence result. From the first n relations of (1.4) we obtain, using (1.2),

$$(-1)^n f_n = B_n f_o - A_n \qquad (1.7)$$

If B_n is non-zero we also have

$$(-1)^n \frac{f_n}{B_n} = f_o - \frac{A_n}{B_n} \qquad (1.8)$$

If we now choose the sequences $\{a_n\}$ and $\{b_n\}$ in such a way that $\exists$ a suffix N such that B_n is non-zero for all $n > N$ then, from result (1.8), a sufficient condition for the continued fraction (1.1) to converge to a solution of the recurrence relations (1.4) is that

$$\lim_{n\to\infty} \frac{f_n}{B_n} = 0 .$$

More particularly, a sufficient condition for convergence is that

$$\lim_{n\to\infty} f_n = 0 \tag{1.9}$$

In this case, if we let a_n and b_n be functions of a complex variable z and if F is the region of the z-plane for which condition (1.9) holds then we can easily prove the following theorem:

Theorem: The continued fraction (1.1) is convergent in that part of the region F which excludes the zeros of $B_n(z)$ for $n > N$, where N is arbitrarily large.

In the remainder of this section we assume that condition (1.9) holds so that the continued fraction (1.1) converges, and we shall call $\{f_r\}$ the corresponding sequence of (1.1).

We now introduce the basic similarity transformation of continued fractions. The values of the continued fraction (1.1) and all its convergents remain unchanged under the transformation

$$f_o = \frac{c_1 a_1}{c_1 b_1 +} \; \frac{c_1 c_2 a_2}{c_2 b_2 +} \; \frac{c_2 c_3 a_3}{c_3 b_3 + \ldots\ldots +} \; \frac{c_{r-1} c_r a_r}{c_r b_r \;\; + \ldots\ldots} \tag{1.10}$$

This is equivalent to multiplying the rth equation of the set (1.4) by γ_r, where $\gamma_r = : \prod_{i=1}^{r} c_i,$ and forming a

New corresponding sequence $\{f'_r\}$, where

$$\left.\begin{array}{l} f'o \;=\; fo \\ f'\, r \;=\; yrfr \end{array}\right\} \qquad (1.11)$$

for r = 1,2,3, ...,

Now, from (1.6) we have the continued fraction

$$\frac{f_n}{f_{n-1}} = \frac{a_{n+1}}{b_{n+1}+}\;\frac{a_{n+2}}{b_{n+2}+}\;\text{..........} \qquad (1.12)$$

for n= 1,2,3, •••• for which we have the following expression, using (1.2),

$$f_o = \frac{A_n + \dfrac{f_n}{f_{n-1}} A_{n-1}}{B_n + \dfrac{f_n}{f_{n-1}} B_{n-1}} \qquad (1.13)$$

for n = 1,2,3, •••••••• • Subtracting the nth convergent of

f_o . and using (1.3) and (1.6) we obtain

$$f_0 - \frac{A_n}{B_n} = \frac{(-1)^n \; \alpha_{n+1}}{B_n (B_{n+1} + \dfrac{f_{n+1}}{f_n} B_n} \qquad (1.14)$$

Hence we have obtained a continued fraction for the truncation error of f_o ,

$$T_n(f_0) \equiv f_0 - \frac{A_n}{B_n} = \frac{(-1)^n \alpha_{n+1}}{B_n B_{n+1} +}\;\frac{a_{n+2} B_n^2}{b_{n+2} +}\;\frac{a_{n+3}}{b_{n+3}}\;\frac{a_{n+4}}{b_{n+4} + \text{.......},} \qquad (1.15)$$

which we shall call the truncation fraction. Also, by comparison with (1.8) we have

$$f_r = \frac{\alpha_{r+1}}{B_{r+1}+} \frac{a_{r+2}B_r}{b_{r+2}+} \frac{a_{r+3}}{b_{r+3}+} \frac{a_{r+4}}{b_{r+4}+ \;.......} \qquad (1.16)$$

The nth denominator of this fraction is B_{r+n} We denote the nth numerator by $A_n^{(r)}$, where $A_n^{(o)}, \equiv A_n, A_1^{(r)} = \alpha_{r+1}$ and

$$A_n^{(r)} = a_{r+n}\, A_{n-2}^{(r)}, + b_{r+n}, A_{n-1}^{(r)} \qquad (1.17)$$

for r = 2,3,4, ... The truncation fraction for f_r is

$$T_n(f_r) \equiv f_r - \frac{A_n^{(r)}}{B_{r+n}} = \frac{(-1)^n \alpha_{r+n+1} B_r}{B_{r+n}B_{r+n+1}+} \frac{a_{r+n+2}B^2_{r+n}}{b_{r+n+2}+} \frac{a_{r+n+3}}{b_{r+n+3}+........,}$$

i.e $\qquad T_n(f_r) = (-1)^n\; B_r\, \dfrac{f_{r+n}}{B_{r+n}}\cdot \qquad (1.18)$

If we now set $f_{r+n} = 0$ then

$$f_0 = \frac{A_{r+n}}{B_{r+n}},\; f_r = \frac{A_n^{(r)}}{B_{r+n}}$$

and (1.3) gives

$$\frac{A_{r+n}}{B_{r+n}} - \frac{A_r}{B_r} = \frac{(-1)^r A_n^{(r)}}{B_r B_{r+n}}$$

Thus we can generalise the determinant formula (1.3) to

$$A_{r+n}B_r - A_rB_{r+n} = (-1)^r A_n^{(r)} \quad . \tag{1.19}$$

Still assuming that condition (1.9) is satisfied we examine a new set of recurrence relations

$$\left.\begin{aligned}
f_1^{(m)} &= \qquad -b_1 f_0^{(m)} \\
f_2^{(m)} &= a_2 f_0^{(m)} - b_2 f_1^{(m)} \\
f_3^{(m)} &= a_3 f_1^{(m)} - b_3 f_2^{(m)} \\
&\qquad \cdots\cdots \\
&\qquad \cdots\cdots \\
f_m^{(m)} &= a_m f_{m-2}^{(m)} - b_m f_{m-1}^{(m)} \\
f_{m+1}^{(m)} &= a_{m+1} f_{m-1}^{(m)} - b_{m+1} f_m^{(m)} + k_{m+1} \\
f_{m+2}^{(m)} &= a_{m+2} f_m^{(m)} - b_{m+2} f_{m+1}^{(m)}
\end{aligned}\right\} \tag{1.20}$$

$$\cdots\cdots$$
$$\cdots\cdots$$

in which the constant term occurs in the (m+l)th relation instead of the first. Apart from the constant term the coefficients are the coefficients of (1.4) and we have, in particular, $k_1 = a_1$ and $f_r^{(o)} \equiv f_r$.

It is easily proved by induction that

$$f_{r-1}^{(m)} = -\frac{B_{r-1}}{B_r} f_r^{(m)} \tag{1.21}$$

for $r = 1,2,3, \ldots m$. In particular, when $r = m$ we substitute for $f_{m-1}^{(m)}$ in the (m+1.)th equation of (1.20)

and obtain

$$f_{m+1}^{(m)} = k_{m+1} - \frac{B_{m+1}}{B_m} f_m^{(m)} \qquad (1.22)$$

Equation (1.22) together with the (m+2)th, (m+3)th, (m+4)th,.... equations of the set (1.20) form a set analogous to (1.4) so

that we obtain the continued fraction

$$f_m^{(m)} = \frac{k_{m+1}}{\left(\frac{B_{m+1}}{B_m}\right)+} \; \frac{a_{m+2}}{b_{m+2}+} \; \frac{a_{m+3}}{b_{m+3}+} \; \cdots\cdots \qquad (1.23)$$

$$= \frac{k_{m+1} B_m}{B_{m+1}+} \; \frac{a_{m+2} B_m}{b_{m+2}+} \; \frac{a_{m+3}}{b_{m+3}+} \; \cdots\cdots \qquad (1.24)$$

using (1.10). In fact we have

$$f_m^{(m)} = \frac{k_{m+1}}{\alpha_{m+1}} B_m f_m \; . \qquad (1.25).$$

By repeated application of (1.21) to (1.25) we have

$$f_r^{(m)} = (-1)^{m-r} \frac{k_{m+1}}{\alpha_{m+1}} B_r f_m \qquad (1.26)$$

for $r \leq m$. Although the continued fraction (1.24) is of

a more convenient form, we must use (1.23) when considering the Corresponding sequence of $f_m^{(m)}$. Applying result (1.16) we get

$$f_r^{(m)} = \frac{k_{m+1}}{\alpha_{m+1}} B_m f_r \tag{1.27}$$

for $r \geq m$.

For results (1.26) and (1.27) we have the truncation

Fractions

$$T_n\left(f_f^{(m)}\right) = (-1)^{m-r} \frac{k_{m+1}}{\alpha_{m+1}} B_r T_n\left(f_m\right)$$

$$= (-1)^{m+n-r} \frac{k_{m+1}}{\alpha_{m+1}} B_r B_m \frac{f_{m+n}}{B_{m+n}} \tag{1.28}$$

for $r \leq m$, and

$$T_n\left(f_r^{(m)}\right) = \frac{k_{m+1}}{\alpha_{m+1}} B_m T_n\left(f_r\right)$$

$$= (1-)^{n} \frac{k_{m+1}}{\alpha_{m+1}} B_r B_m \frac{f_{r+n}}{B_{r+n}} \tag{1.29}$$

for $r \geq m$.

Finally, we state some results whose usefulness will become apparent in the next section. Analogous to (1.10), we can transform the set (1.20) to a more convenient form, constructing a new corresponding sequence $\left\{f_r^{(m)}\right\}$ where

$$f_0^{(m)'} = f_0^{(m)}$$
$$f_r^{(m)'} = y_r f_0^{(m)} \qquad (1.30)$$

and a new constant term k_{m+1} where

$$K'_{m+1} = Y_{m+1} \quad k_{m+1} \qquad (1.31)$$

Also useful are the determinantal forms for the numerators and denominators of the continued fraction (1.1) There are :

$$A_n = a_1 \begin{vmatrix} b_2 & 1 & & & & \\ -a_3 & b_3 & 1 & & & \\ & -a_4 & b_4 & 1 & & \\ & & \cdots & \cdots & & \\ & & & \cdots & \cdots & 1 \\ & & & & -a_n & b_n \end{vmatrix} \qquad (1.32)$$

and

$$B_n = \begin{vmatrix} b_1 & 1 & & & & \\ -a_2 & b_2 & 1 & & & \\ & -a_3 & b_3 & 1 & & \\ & & \cdots & \cdots & & \\ & & & \cdots & \cdots & 1 \\ & & & & -a_n & b_n \end{vmatrix} \qquad (1.33)$$

2 Application to General Linear Birth-Death Processes

The following set of differential-difference equations represent a general linear birth-death process:

$$\left.\begin{aligned} P_O'(t) &= -\lambda_o p_o(t) + \mu_1 P_1(t) \\ P_r'(t) &= \lambda_{r-1}\, p_{r-1}(t) - (\lambda_r + \mu_r)\, P_r(t) + \mu_{r+1}\, P_{r+1}(t) \end{aligned}\right\} \qquad (2.1)$$

for $r = 1,2,3,$.-.. and where $0 \leq p_r(t) \leq 1$ and

$\sum_{r=0}^{\infty} Pr(t) = 1$, subject to the initial conditions

$$P_r(0) = \delta_{r,m} \qquad (2.2)$$

for some $m \varepsilon \{ 0,1,2, \ldots]$. Also $\lambda_r > 0$ for $r = 0,1,2, \ldots$

and $\mu_r > 0$ for $r = 1,2,3, \ldots$ and we define

$$Lr = \sum_{i=0}^{r} \lambda i \;, \quad Mr = \sum_{i=1}^{r} \mu i \;,$$

and $L_{-1} = M_o = 1$.

The set of equations (2.1) has been solved analytically, in a few particular cases, by a generating function method but the set may be solved numerically in the general case using the results of section 1. However, a limiting factor for the numerical solution is the working accuracy of the computer used.

We denote the Laplace transform of $p_r(t)$ by $D_r(s)$ where

$$P_r(s) = \int_0^{\infty} e^{-st} \, p_r(t) \, dt \qquad (2.3)$$

Laplace transforming (2.1) and rearranging we have

$$\left.\begin{aligned} P_1 &= -\frac{\delta_{o,m}}{\mu_1} - \left(-\frac{\lambda_{0+s}}{\mu_1}\right) P_0 \\ P_{r+1} &= -\frac{\lambda_{r-1}}{\mu_{r+1}} p_{r-1} - \left(-\frac{\lambda_r + \mu_{r+s}}{\mu_{r+1}}\right) P_r - \frac{\delta_{r,m}}{\mu_{r+1}} \end{aligned}\right\} \quad (2.4.)$$

The set (2.4) is now of the form (1.20). However, to convert the resultant continued fraction to a convenient form we apply the transformations (1.30) and (1.31) using $yr = (-1)^r M_r$ The set (2.4) then becomes

$$\left.\begin{aligned} f_1^{(m)} &= \delta_{o,m} - (\lambda_0 + s)\, f_0^{(m)} \\ f_{r+1}^{(m)} &= -\lambda r{-1}\, \mu_r\, f_{r-1}^{(m)} - (\lambda_r + \mu_r + s)\, f_r^{(m)} + (-1)^m M_m \delta_{r,m} \end{aligned}\right\} \quad (2.5)$$

where $P_O = f_O^{(m)}$ and

$$p_r = \frac{(-1)^r}{M_r} f_r^{(m)} \qquad (2.6)$$

for r = 1,2,3, ...we now have the continued fraction

$$f_0 = \frac{1}{\lambda_0 + s -} \frac{\lambda_0 \mu_1}{\lambda_1 + \mu_1 + s -} \frac{\lambda_1 \mu_2}{\lambda_2 + \mu_2 + s -} \cdots\cdots \frac{\lambda_{r-1} \mu_r}{\lambda_r + \mu_r + s} \cdots\cdots\cdots . \qquad (2.7)$$

Since the population cannot grow to infinite size in finite time we have, for finite t,

$$\lim_{r\to\infty} p_r(t) = 0$$

So we have, using (2,6) and (2,3),

$$\lim_{r\to\infty} f_r = (-1)^r M_r \int_0^\infty e^{-st} \left\{ \lim_{r\to\infty} P_r(t) \right\} dt = 0 .$$

Hence the region F is the whole s-plane and we may apply the convergence theorem of section 1 . if we can find the positions of the zeros of the denominators of the continued fraction (2.7). From (1.33) we have

$$B_n = \begin{vmatrix} \lambda_0 + s & 1 & & & & \\ \lambda_0\mu_1 & \lambda_1 + \mu_1 + s & 1 & & & \\ & \lambda_1\mu_2 & \lambda_2 + \mu_2 + s & 1 & & \\ & & & \cdots\cdots & & \\ & & & \cdots\cdots & & \\ & & & & & 1 \\ & & & & \lambda_{n-2}\mu_{n-1} & \lambda_{n-2}\mu_{n-1} + s \end{vmatrix}$$

which is clearly zero when -s is an eigenvalue of the matrix

$$C_n = \begin{vmatrix} \lambda_0 + s & 1 & & & & \\ \lambda_0\mu_1 & \lambda_1 + \mu_1 & 1 & & & \\ & \lambda_1\mu_2 & \lambda_2 + \mu_2 & 1 & & \\ & & & \cdots\cdots & & \\ & & & \cdots\cdots & & \\ & & & & & 1 \\ & & & & \lambda_{n-2}\mu_{n-1} & \lambda_{n-2}\mu_{n-1} \end{vmatrix}$$

This matrix is quasi-symmetric and may be transformed into a real symmetric matrix by a similarity transformatio

$$E_n = D_n^{-1} C_n D_n$$

where the matrix $D_n = \text{diag}\left\{1, \sqrt{L_0M_1}, \sqrt{L_1M_2}, \ldots \sqrt{L_{n-2}M_{n-1}}\right\}$

The matrix so formed is

$$E_n = \begin{bmatrix} \lambda_0 & \sqrt{\lambda_0\mu_1} & & & \\ \sqrt{\lambda_0\mu_1} & \lambda_1+\mu_1 & \sqrt{\lambda_1\mu_2} & & \\ & \sqrt{\lambda_1\mu_2} & \lambda_2\mu_2 & \sqrt{\lambda_2\mu_3} & \\ & & & \cdots\cdots\cdots\cdot & \\ & & & \cdots\cdots\cdots\cdot\ \cdot\cdot & \\ & & & & \sqrt{\lambda_{n-2}\mu_{n-1}} \\ & & & \sqrt{\lambda_{n-1}\mu_{n-1}} & \lambda_{n-1}+\mu_{n-1} \end{bmatrix}$$

The matrix E_n is a real symmetric positive definite tridiagonal matrix with non-zero subdiagonal elements. Because of these properties the eigenvalues are real, positive and distinct, [See Wilkinson (1965).] Hence B_n (s) has only simple zeros which all lie on the negative real axis in the s-plane and, from the theorem of section 1., we can state that the continued fraction (2.7) converges in the s-plane cut from 0 to ∞ along the negative real axis.
We are now justified in using the results (l.26) and (1.27) to give the following expressions for P_r (s) :

$$P_r = \frac{(-1)^m}{L_{m-1}\,M_r} B_r\, f_m \qquad (2.8)$$

(2.8)

for $r \le m$, and

$$P_r = \frac{(-1)^r}{L_{m-1}\,M_r} B_m\, f_r \qquad (2.9)$$

for $r \ge m$. Writing $P_{r,n}$ for the nth convergent of P_r (s) and using (1.16) we have

$$P_{r,n} = \frac{(-1)^m}{L_{m-1} M_r} B_r \frac{An^{(m)}}{B_{m+n}} \tag{2.10}$$

for $r \leq m$, and

$$P_{r,n} = \frac{(-1)^r}{L_{m-1} M_r} B_m \frac{A_n^{(r)}}{B_{r+n}} \tag{2.11}$$

for $r \geq m$. wee are also justified in inverting the $\mathcal{L}$-transform expressions (2.10) and (2.11) since all the singularities of $P_{r,n}$ lie to the left of the imaginary axis in the s-plane. In general we consider a convergent K(s) such that

$$K(s) = \frac{N(s)}{B_n(s)} \tag{2.12}$$

where B_n (s) is a denominator polynomial of order n in s and N(s) is the numerator polynomial which is of lower order. If we choose $-z_1$, $-z_2$, ... $-z_n$ to be the real, negative and distinct roots of B_n (s) then we can write

$$B_n(s) = \prod_{i=1}^{n} (s + z_i) \tag{2.13}$$

Since the roots are distinct we may write K(s) in the partial fraction form

$$K(s) \sum_{i=1}^{n} \frac{\omega_i}{s+z_i} \tag{2.14}$$

where $\omega_1, \omega_2, \ldots \omega_n$ are constants given by

$$\omega_i = \frac{N(-z_i)}{B_n'(-z_i)} \tag{2.15}$$

and where $B_n'(-z_i)$ is computed from

$$B_n(-z_i) = \prod_{\substack{j=i \\ i \neq J}}^{n} (z_j - z_i). \tag{2.16}$$

Inverting, we have the solution

$$\mathscr{L}^{-1} k(s) = \sum_{i=1}^{n} \omega_i e - z_i t \tag{2.17}$$

which is the form in which the probabilities,

$D_r(t)$, are computed.

To greatly reduce the required computation, since we only require the values of' $A_n^{(r)}$ at the roots of B_{r+n}, we appeal to the generalised determinant formula (1.19). From this we get that, at a root of B_{r+n},

$$A_n^{(r)} = (-1)^r A_{r+n} B_r \tag{2.18}$$

Hence we need only compute the roots of the numerators and denominators of the continued fraction (2.7) in order to compute the probabilities, $p_r(t)$, for apy value of m. The roots of the numerators are also computed as eigenvalues using (1.32).

From (1.28) and (1.29) we have the truncation results

$$T_n(P_r) = \frac{(-1)^{m+n}}{L_{m-1}M_r} B_r B_m \frac{f_{m+n}}{B_{m+n}} \tag{2.19}$$

for $r \leq m$, and

$$T_n(P_r) = \frac{(-1)^{r+n}}{L_{m-1}\, m_r} B_r\, B_m \frac{F_{r+n}}{B_{r+n}} \tag{2.20}$$

for $r \geq m$.

We will now derive estimates of the truncation errors in the probabilities obtained from results (2.10) and (2.11). We observe from (2.7) that for |s| large,

$$B_n(s) = (\lambda_o + s)(\lambda_1 + \mu_1 + s)(\lambda_2 + \mu_2 + s)....(\lambda_{n-1} + \mu_{n-1} + s) + 0(s^n{-}^2) \tag{2.21}$$

for n = 2,3,4, ... and also, from (1.16),

$$f_n = \frac{(-1)^n L_{n-1} M_n}{(\lambda_0 + s)(\lambda_1 + \mu_1 + s)...(\lambda_n + \mu_n + s) + 0(s^{n-1})} \tag{2.22}$$

for |s| large and n= 1 ,2,3, •••• •

we define

$$\sigma_n = \lambda_0 + \sum_{r=1}^{n-1} (\lambda_r + \mu_r)$$

so that, for $|s|$ large, (2.19) may be written

$$P_r - P_{r,n} = \frac{L_{m+n-1} M_{m-+n}}{L_{m-1} M_r} \frac{1}{3^{2n+m-r+1}} \left\{ 1 - \frac{\sigma_{m+n} + \sigma_{m+n+1} - \sigma_m - \sigma_r + 0\left(\frac{1}{s^2}\right)}{S} \right\}$$

for $r \leq m$. Inverting, we obtain, for t small

$$P_r(t) - \mathcal{L}^{-1} \left\{ P_{r,n} \right\} = \frac{L_{m+n-1} M_{m+n}}{L_{m-1} M_r} \frac{t^{2n+m-r}}{(2n+m-r)!} \left\{ 1 - \frac{\sigma_{m+n} + \sigma_{m+n+1} - \sigma_m - \sigma_r}{2n+m-r+1} t + 0(t^2) \right\}$$

(2.23)

for $r \leq m$. In (2.23) the dominant term provides an upper bound which is only a useful error estimate if n is large. We find, however, that for moderate n a satisfactory estimate is obtained choosing an unbounded function which agrees with the first two terms of (2.23). We choose

$$P_r(t) - \mathcal{L}^{-1} P_{r,n} = \frac{L_{m+n-1} M_{m+n}}{L_{m-1} M_r} \frac{t2_{n+m-r}}{2_{n+m-r!}} \left\{ \frac{1}{(1 + P_{r,m+n} t) 2n + m - r - 1} + 0\left(t^2\right) \right\} \qquad (2.24)$$

for $r \leq m$ where

$$P_{r,m+n} = \frac{\sigma_{m+n} \sigma_{m+n+1} - \sigma_m - \sigma_r}{(2n + m - r + 1)(2n + mr - 1)}$$

From (2.20) We also have

$$P_r(t) - \mathcal{L}^{-1}\left\{P_{r,n}\right\} = \frac{L_{m+n-n}M_{m+n}}{L_{m-1}M_r}\frac{t^{2n+r-m}}{(2n+r-m)!}\left\{\frac{1}{(1+P_{m,r+n}t)^{2n+r-m-1}} + 0(t^2)\right\}$$

For $r \geq m$ (2.25)

Given a value or n and a sufficiently small error $\in$ the results (2.24) and (2.25) may be used to estimate a range of t for which this error is not exceeded. A larger value of $\in$ could give a very pessimis tic estimate for the range of t.

Examples of Birth-Death Processes

We conclude with numerical results for four examples of linear birth-death processes. The models we use are

(i) An immigration-death process with $\lambda_n = 0.2$ and $\mu_n = 0.4n$ for n = 0,1,2,3,.....................For this model the probabilities tend to steady state values. The results are evaluated in the two cases when the initial population size m is 0 and 1.

(ii) Erlang's model with $\lambda_n = 0.4$ for n =0,1,2,3, ..., $\mu_o = 0$ and $\mu_n = 0.2$ for n = 1,2,3,.................... In this case there are no steady state values. These results are evaluated when m = 0 and when m = 5.

(iii) A three-server queuing model with $\lambda_n = 0.6$ for n = 0,1,2,3, . . •, $\mu_o = 0$, $\mu_1 = \mu_2 = 0.2$, $\mu_3 = \mu_4 = 0.4$ and $\mu_n = 0.6$ for n = 5,6,7, ...• This represents a queuing system in which the number of servers is dependent on queue size. We choose m = 0.

(iv) A process with $\lambda_n = 0.3$ and $\mu_N = 0.1\sqrt{n}$ for $n = 0,1,2,3, \ldots$. Again, we choose m = 0 .

Analytic solutions for models (i) and (ii) may be obtained by the generating function method.

The table below contains estimates of ranges of t for selected values of n using the formulae (2.24) and (2.25). In each case 10^{-4} is the chosen maximum error in the computed value of $p_r(t)$.

Model	m	r	n	Estimated Range (to 2 sig.figs,)
(i)	0	0	5	$0 \leq t \leq 6.9$
	0	0	10	$0 \leq t \leq 60$
	1	1	10	$0 \leq t \leq 72$
(ii)	0	0	5	$0 \leq t \leq 9.8$
	0	0	10	$0 \leq t \leq 39$
	5	5	10	$0 \leq t \leq 40$
(iii)	0	0	5	$0 \leq t \leq 6.2$
	0	0	10	$0 \leq t \leq 54$
(iv)	0	0	5	$0 \leq t \leq 12.5$
	0	0	10	$0 \leq t \leq 38$

In FIGS 1.- 6.all results were computed with n = 10 using the range $0 \leq t \leq 40$, As a check the results were recomputed

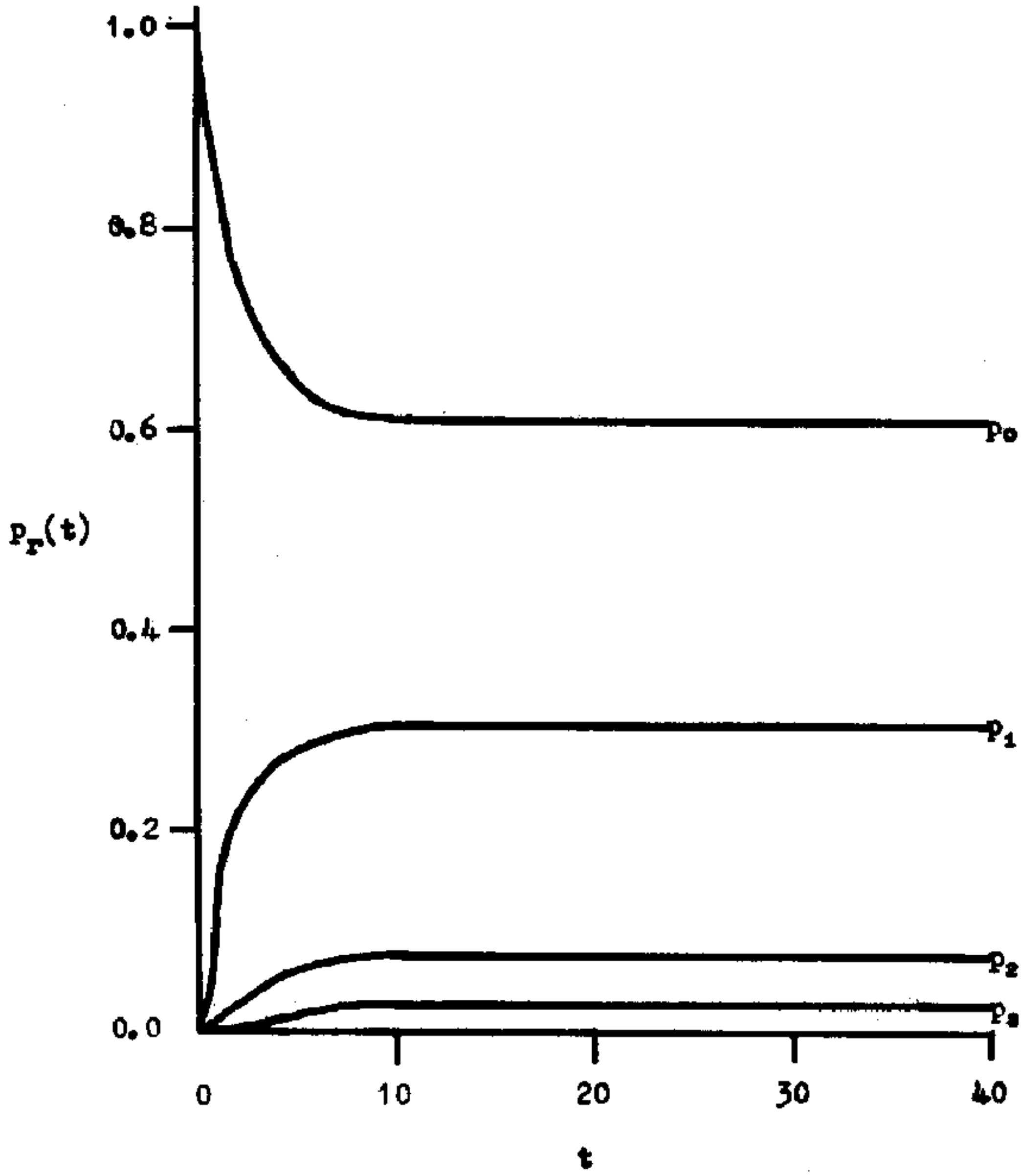

FIG. 1. Model (i) with m=0.

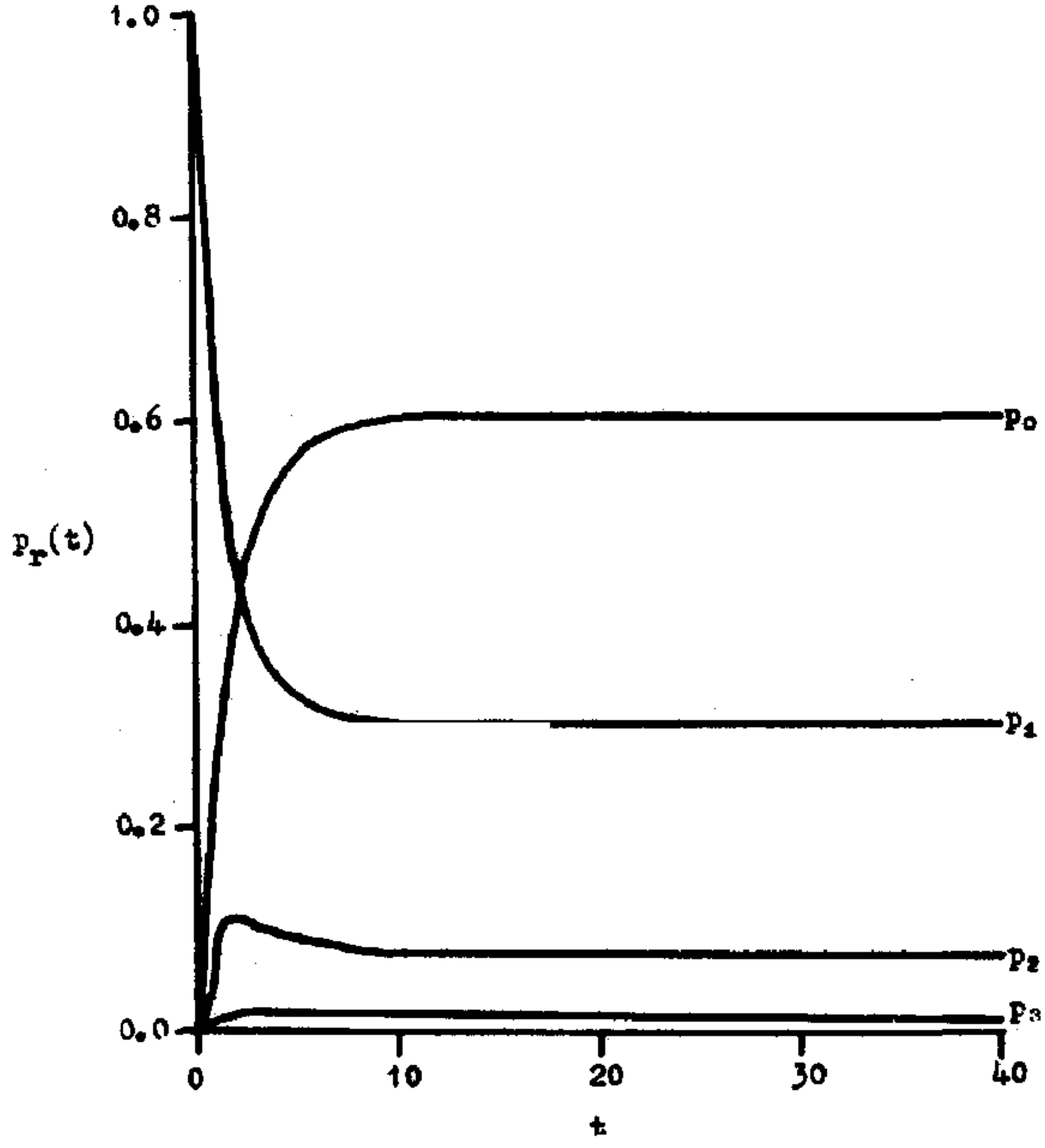

FIG. 2. Model (i) with m=1.

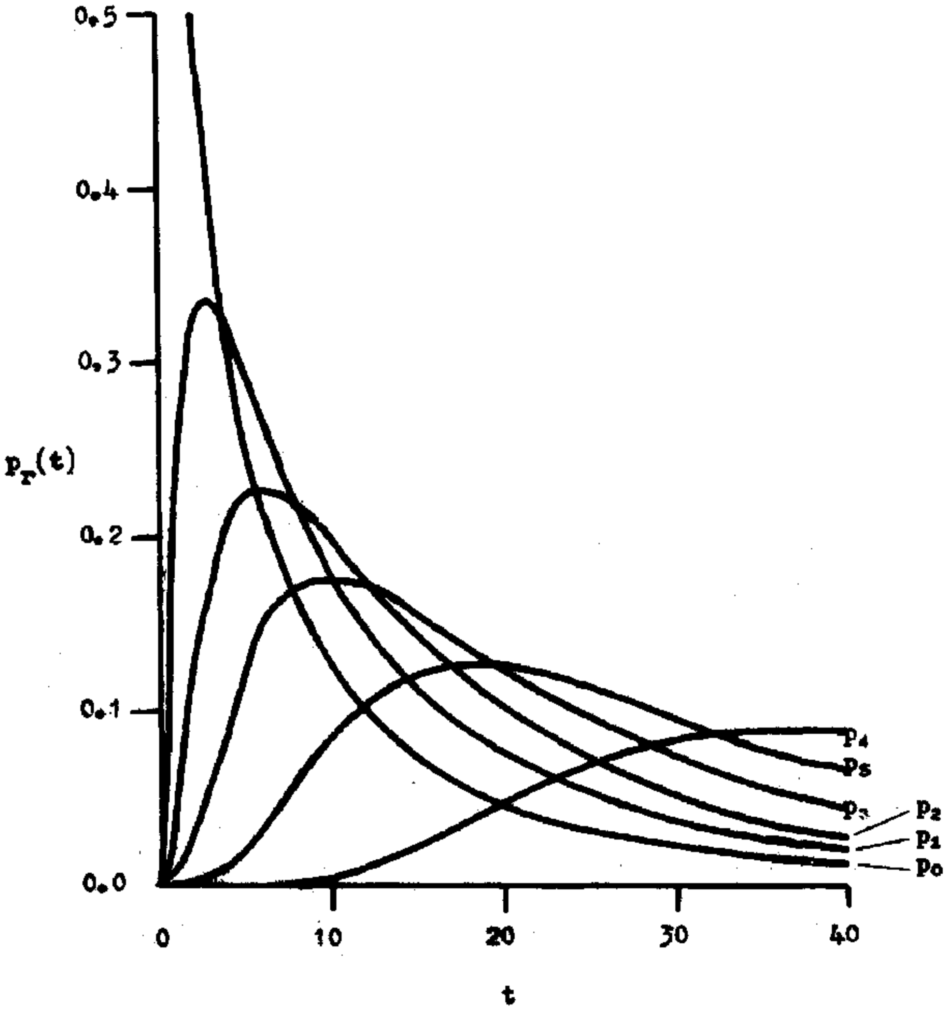

FIG. 3. Model (ii) with m=0.

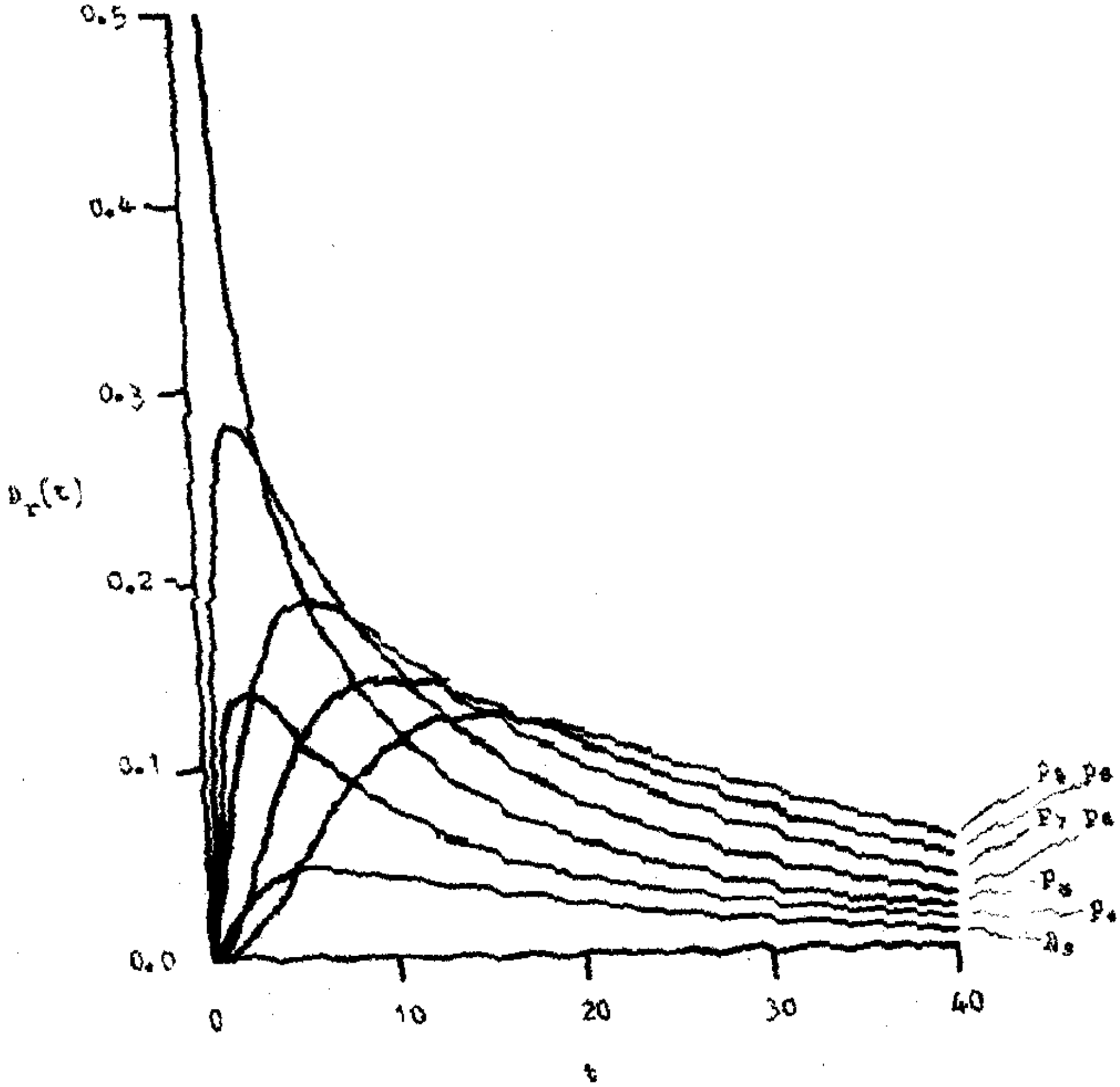

FIG. 4. Model (11) with m=5

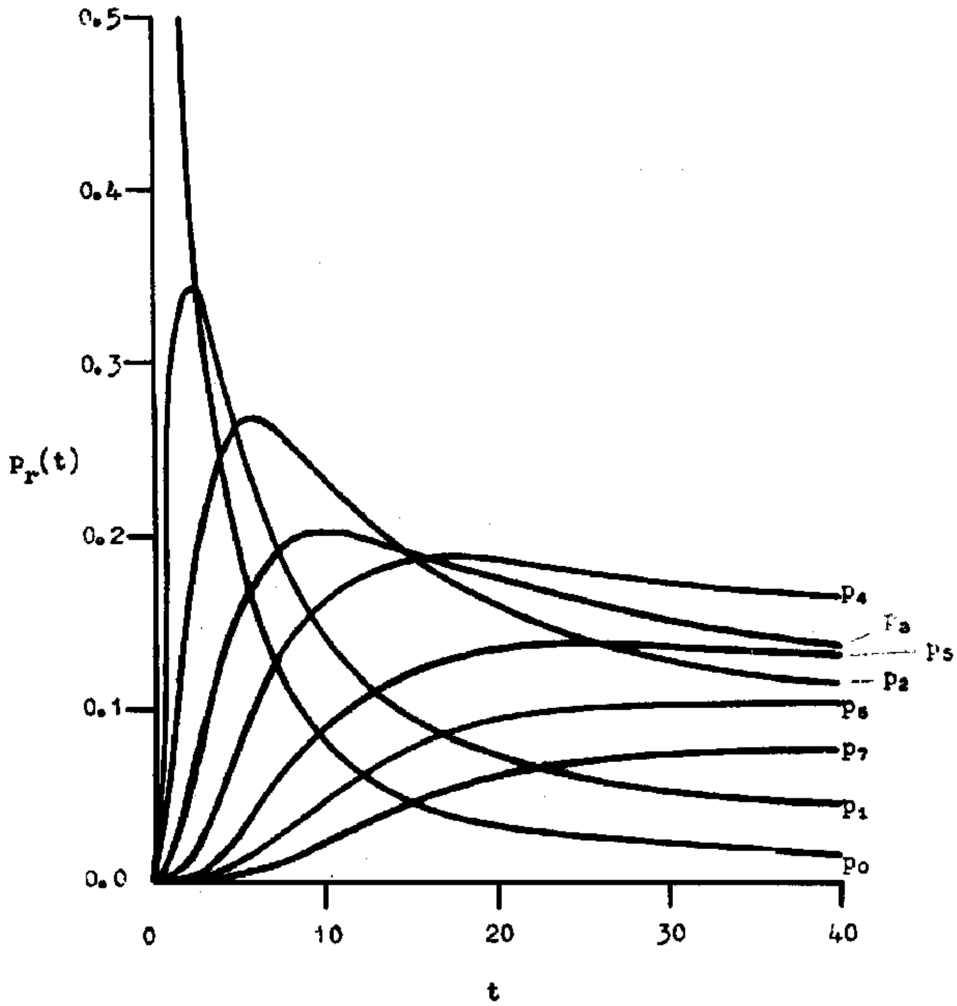

FIG. 5. Model (iii) with m=0.

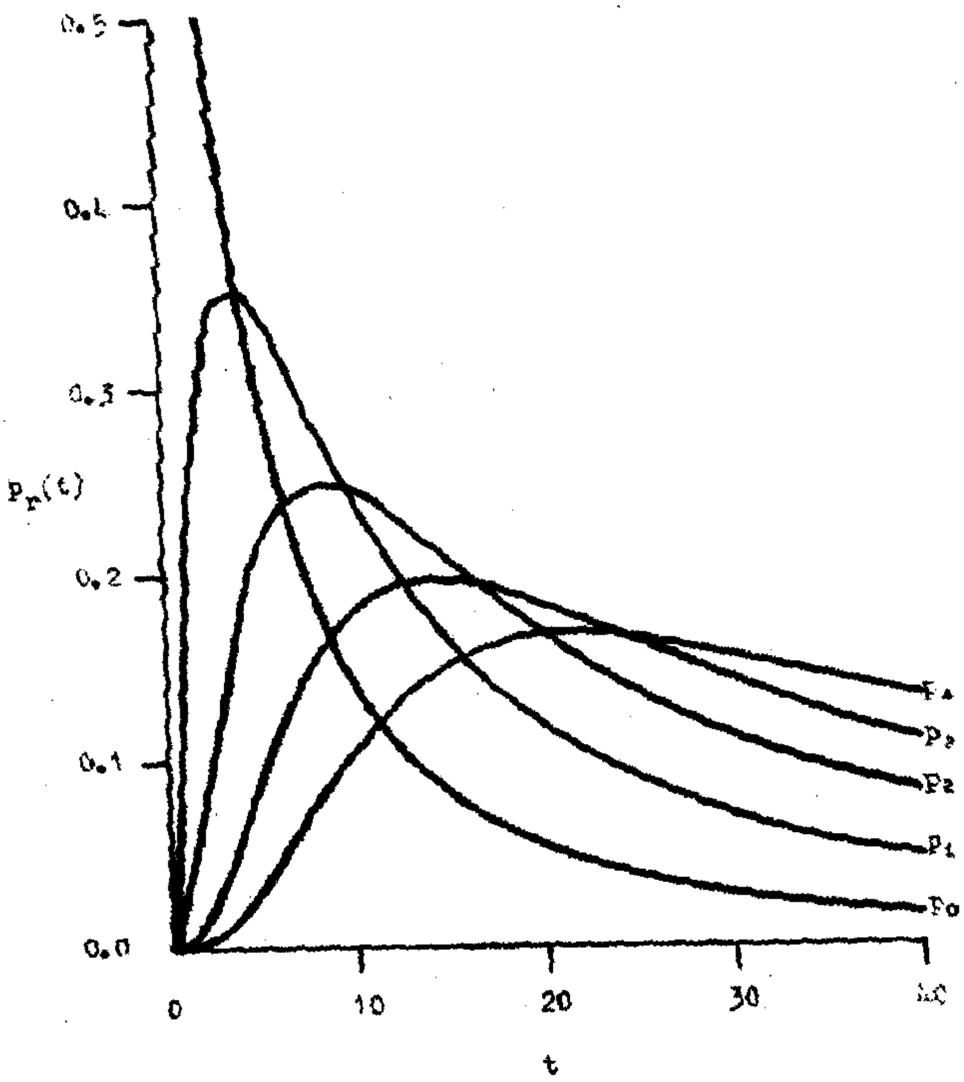

FIG. 6. Model (iv) with m=0.

with n = 15 and the range estimates in the above table were all found to be smaller than the actual range for the chosen accuracy.

The eigenvalues of the matrix E_n were computed using an algorithm based on that given by Bowdler, et.al.(1968). It was, however, found necessary to compute these eigenvalues using an accuracy of about 20 significant figures because some of the calculations are ill-conditioned Finally, the only serious drawback of the method is that it is limited by the size and working accuracy of the computer used so that efficient programming is essential.

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