Title Quadrature Filters for One-step Randomly Delayed Measurements

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Abstract

In this paper, two existing quadrature filters, *viz.*, the Gauss-Hermite filter (GHF) and the sparse-grid Gauss-Hermite filter (SGHF) are extended to solve nonlinear filtering problems with one step randomly delayed measurements. The developed filters are applied to solve a maneuvering target tracking problem with one step randomly delayed measurements. Simulation results demonstrate the enhanced accuracy of the proposed delayed filters compared to the delayed cubature Kalman filter and delayed unscented Kalman filter.

Keywords - Nonlinear filtering, Randomly delayed measurements, Gauss-Hermite quadrature rule, Product rule, Smolyak rule.

1 Introduction

The estimation of states of a dynamic system is necessary to solve problems in many active research areas, *viz.* target tracking [1], economics and finance [2,3], wireless communication [4], traffic management [5], to name a few. For a linear system with Gaussian noises, optimal solution exists and is popularly known as Kalman filter. However, for nonlinear systems, no optimal solution is available. As most of the practical systems are nonlinear, it becomes necessary to develop a more accurate and computationally efficient filter to solve nonlinear estimation problems. In order to achieve higher estimation accuracy with reasonable computational load, several nonlinear filters have been introduced. These include the extended Kalman filter (EKF) [6], the unscented Kalman filter (UKF) [7] and its variants [8], the cubature Kalman filter (CKF) [9, 10], the Gauss-Hermite filter (GHF) [11, 12], the sparse-grid Gauss-Hermite filter (SGHF) [13], the central difference filter (CDF) [14], the divided difference filter (DDF) [15] *etc.*

For all the above mentioned filters, it is assumed that the current measurements are available at every time instant. However in practice, situations may arise where the measurement data arrives at the sensor with some random delay or measurements may be randomly delayed from the sensors to the filter as a result of limited communication bandwidth. In the literature, the problem appears with different names such as out of sequence measurement (OOSM) [16], filtering with random sample delay [17], filtering with random time delayed measurements [18] *etc.*

The literature available on state estimation with non-delayed measurements is rich. However, the same is not true for randomly delayed measurement problems except a few notable publications on linear systems [18–21], and nonlinear systems [22–24]. The literature on the described problem began with the work of Ray *et al.*, where the authors developed a randomly delayed filtering method for the linear systems. Later, Carazo and others introduced nonlinear filtering algorithm for one time step [22] and two time step [23] randomly delayed measurements using the extended and the unscented Kalman filter approach. Recently, Wang *et al.* [24] incorporated cubature Kalman filter (CKF) [9] to solve the nonlinear filtering problem with one-step randomly delayed measurements.

In the present work, initially GHF is extended to solve the nonlinear filtering problems with one step randomly delayed measurements. We abbreviate the new filter as GHF-1RD. Accuracy of GHF-1RD is high, although the computational load increases exponentially with the dimension of the system. To circumvent the problem, one step randomly delayed SGHF (SGHF-1RD) has been developed in this paper. The proposed delay filters have been applied to a maneuvering target tracking problem. The same problem was discussed in the context of delay-free filters previously in [25]. In this paper, the chances of delays are incorporated and the developed delayed filters are implemented for tracking. The simulation results show that the GHF-1RD provides better accuracy compared to its UKF and CKF counterparts. The SGHF-1RD provides a similar accuracy as GHF-1RD with much lower computational cost.

2 Problem formulation

Let us consider a discrete nonlinear system with a state equation,

$$\mathbf{x}_k = \phi_{k-1}(\mathbf{x}_{k-1}) + q_{k-1}, \tag{1}$$

and a measurement equation,

$$z_k = \gamma_k(\mathbf{x}_k) + w_k, \tag{2}$$

where $\mathbf{x}_k \in \Re^n$ denotes the state vector of a system, and $z_k \in \Re^p$ is the measurement at any instant k, where $k = \{1, 2, 3, ...\}$. $\phi_k(\mathbf{x}_k)$ and $\gamma_k(\mathbf{x}_k)$ are known nonlinear functions of $\mathbf{x}_k$ and k. The process noise $q_k \in \Re^n$ and the measurement noise $w_k \in \Re^p$ are assumed to be uncorrelated, white, and following Gaussian distribution with covariance Q_k, and R_k respectively.

It is assumed that the measurement is one step randomly delayed, *i.e.*, at any instant of time the received measurement may be the measurement of that instant or one step earlier. So one step randomly delayed measurement equation could be written as

$$y_k = (1 - \beta_k)z_k + \beta_k z_{k-1}, \quad y_1 = z_1, \tag{3}$$

where β_k is mutually independent of the process and measurement noises q_k and w_k, and a sequence of Bernoulli random variable. It may take values either 0 or 1, with the probability

$$\begin{aligned} &P(\beta_k = 1) = p_k = E[\beta_k], \\ &P(\beta_k = 0) = 1 - p_k, \\ &E[(\beta_k - p_k)^2] = p_k(1 - p_k). \end{aligned}$$

$\beta_k = 0$ implies that the current measurement has arrived (no delay measurement), while $\beta_k = 1$ means that the previous step's measurement has arrived (one step delayed measurement).

The objective here is to design a nonlinear filter for the system described by equations (1)-(3). More specifically, posterior pdf $P(\mathbf{x}_k|Y_k)$, where $Y_k = \{y_i\}$ with $i = \{1, 2, ..., k\}$ denotes the set of the delayed measurements, needs to be determined.

To solve the above mentioned problem, we propose two new estimators in this paper, namely Gauss-Hermite filter with one step randomly delayed measurement (GHF-1RD) and sparse Gauss-Hermite filter with one step randomly delayed measurement (SGHF-1RD). For an arbitrary nonlinear system, the posterior probability density function (pdf), $P(\mathbf{x}_k|Y_k)$, will no longer be Gaussian. However, we approximate it with a Gaussian distribution which is characterized by posterior mean $\hat{\mathbf{x}}_{k|k}$ and covariance $\mathbf{P}_{k|k}$. This is a standard assumption in the literature on filters based on numerical integration such as GHF and SGHF, see, e.g. [11] and [13]. The proposed filters estimate the posterior mean and the covariance at each step recursively.

Through simulation experiments, we show that the proposed methods perform better compared to the existing UKF and CKF based filters for single step randomly delayed measurements. GHF-1RD uses product rule to generate support points in multidimensional space. So the support point requirement for GHF-1RD increases exponentially with the dimension of the system, hence suffers from the *curse of dimensionality* problem. The said problem is overcome in the SGHF-1RD which uses Smolyak rule, introduced in [13, 26] to extend the single dimensional quadrature points to multidimensional space without affecting estimation accuracy achieved in GHF-1RD.

3 Filtering with one step randomly delayed measurements under Bayesian framework

The measurement equation, where the measurements are randomly delayed by one sampling time, could be written as

$$\begin{aligned} y_k &= (1-\beta_k)z_k + \beta_k z_{k-1} \\ &= (1-\beta_k)(\gamma_k(\mathbf{x}_k) + w_k) + \beta_k(\gamma_{k-1}(\mathbf{x}_{k-1}) + w_{k-1}). \end{aligned}$$

To estimate the states, $P(w_k|Y_k)$ needs to be calculated along with $P(\mathbf{x}_k|Y_k)$. Hence, the state vector is augmented with measurement noise ($\mathbf{x}^a_{k+1} = [\mathbf{x}^T_{k+1} \quad w^T_{k+1}]^T$), and we estimate the posterior pdf of augmented states $P(\mathbf{x}^a_k|Y_k)$. Here superscript 'a' stands to represent the augmentation.

3.1 Assumptions

The probability densities at each time step in general are non-Gaussian in nature. The quadrature filters assume them as Gaussian and approximate with first and second order moments. We assume the following:

- The probability density function, $P(\mathbf{x}_{k+1}|Y_{k+1})$, is Gaussian with mean $\hat{\mathbf{x}}_{k+1|k+1}$, and covariance $\mathbf{P}_{k+1|k+1}$.
- The probability density function, $P(w_{k+1}|Y_{k+1})$ is Gaussian with mean $\hat{w}_{k+1|k+1}$ and covariance $\mathbf{P}^{ww}_{k+1|k+1}$.
- The pdf of augmented states follows Gaussian distribution, *i.e.*,

  $$P(\mathbf{x}^a_{k+1}|Y_{k+1}) = \aleph(\mathbf{x}^a_{k+1}; \hat{\mathbf{x}}^a_{k+1|k+1}, \mathbf{P}^a_{k+1|k+1})$$

  where $\aleph(x;\mu,\Sigma)$ represents the Gaussian distribution of 'x' with mean μ and covariance Σ, and

  $$\mathbf{x}^a_{k+1|k+1} = \begin{bmatrix} \hat{\mathbf{x}}_{k+1|k+1} \\ \hat{w}_{k+1|k+1} \end{bmatrix},$$

  and

  $$\mathbf{P}^a_{k+1|k+1} = \begin{bmatrix} \mathbf{P}_{k+1|k+1} & \mathbf{P}^{\mathbf{x}w}_{k+1|k+1} \\ (\mathbf{P}^{\mathbf{x}w}_{k+1|k+1})^T & \mathbf{P}^{ww}_{k+1|k+1} \end{bmatrix},$$

  with $\hat{w}_{k+1|k+1}$ being the posterior estimate for measurement noise, while $\mathbf{P}^{ww}_{k+1|k+1}$ and $\mathbf{P}^{\mathbf{x}w}_{k+1|k+1}$ being the noise covariance and cross-covariance between the state and measurement noise respectively.
- The one-step predictive pdf of $\mathbf{x}_{k+1}$ is Gaussian:

  $$P(\mathbf{x}_{k+1}|Y_k) = \aleph(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}_{k+1|k})$$

- The one-step predictive pdf of delayed measurement y_{k+1} is Gaussian, *i.e.*

  $$P(y_{k+1}|Y_k) = \aleph(y_{k+1}; \hat{y}_{k+1|k}, \mathbf{P}^{yy}_{k+1|k}).$$

- The pdf of non-delayed measurement, $P(z_k|Y_k)$, and $P(z_{k+1}|Y_k)$ are Gaussian with mean $\hat{z}_{k|k}$, $\hat{z}_{k+1|k}$ and covariance $\mathbf{P}^{zz}_{k|k}$, and $\mathbf{P}^{zz}_{k+1|k}$ respectively.

The estimation of the augmented states can be done in two steps [13], namely (i) measurement noise estimation (ii) state estimation.

3.2 Measurement noise estimation

The first and second order moments of prior and posterior pdf of measurements are given by

$$\hat{z}_{k+1|k} = \int \gamma_{k+1}(\mathbf{x}_{k+1})\aleph(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}_{k+1|k})d\mathbf{x}_{k+1} \tag{4}$$

$$\mathbf{P}^{zz}_{k+1|k} = \int \gamma_{k+1}(\mathbf{x}_{k+1})\gamma^T_{k+1}(\mathbf{x}_{k+1})\aleph(\mathbf{x}_{k+1}; \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}_{k+1|k})d\mathbf{x}_{k+1} - \hat{z}_{k+1|k}\hat{z}^T_{k+1|k} + R_{k+1} \tag{5}$$

$$\hat{z}_{k|k} = \int [\gamma_k(\mathbf{x}_k) + w_k]\aleph(\mathbf{x}^a_k; \hat{\mathbf{x}}^a_{k|k}, \mathbf{P}^a_{k|k})d\mathbf{x}^a_k \tag{6}$$

$$\mathbf{P}^{zz}_{k|k} = \int [\gamma_k(\mathbf{x}_k) + w_k][\gamma_k(\mathbf{x}_k) + w_k]^T\aleph(\mathbf{x}^a_k; \hat{\mathbf{x}}^a_{k|k}, \mathbf{P}^a_{k|k})d\mathbf{x}^a_k - \hat{z}_{k|k}\hat{z}^T_{k|k} \tag{7}$$

The predicted mean and error covariance of the delayed measurements are given by:

$$\hat{y}_{k+1|k} = (1 - p_{k+1})\hat{z}_{k+1|k} + p_{k+1}\hat{z}_{k|k}$$

$$\mathbf{P}^{yy}_{k+1|k} = (1 - p_{k+1})\mathbf{P}^{zz}_{k+1|k} + p_{k+1}\mathbf{P}^{zz}_{k|k} + p_{k+1}(1 - p_{k+1})(\hat{z}_{k+1|k} - \hat{z}_{k|k})(\hat{z}_{k+1|k} - \hat{z}_{k|k})^T$$

The Kalman gain for noise is

$$K^w_k = \mathbf{P}^{wy}_{k+1|k}(\mathbf{P}^{yy}_{k+1|k})^{-1},$$

where the cross error covariance is given by

$$\mathbf{P}^{wy}_{k+1|k} = (1 - p_{k+1})R_{k+1}.$$

Finally, the expressions for the posterior estimate of the measurement noise, and its error covariance are

$$\hat{w}_{k+1|k+1} = K^w_k(y_{k+1} - \hat{y}_{k+1|k}),$$

and

$$\mathbf{P}^{ww}_{k+1|k+1} = R_{k+1} - K^w_k\mathbf{P}^{yy}_{k+1|k}(K^w_k)^T.$$

3.3 State estimation

Prior estimated state and its covariance can be written as

$$\hat{\mathbf{x}}_{k+1|k} = \int \phi_k(\mathbf{x}_k)\aleph(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k}, \mathbf{P}_{k|k})d\mathbf{x}_k \tag{8}$$

$$\mathbf{P}_{k+1|k} = \int \phi_k(\mathbf{x}_k)\phi^T_k(\mathbf{x}_k)\aleph(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k}, \mathbf{P}_{k|k})d\mathbf{x}_k - \hat{\mathbf{x}}_{k+1|k}\hat{\mathbf{x}}^T_{k+1|k} + Q_k \tag{9}$$

The cross error covariances are given by

$$\begin{aligned}\mathbf{P}^{\mathbf{x}z}_{k+1,k|k} &= E[\tilde{\mathbf{x}}_{k+1|k}\tilde{z}^T_{k|k}|Y_k] \\ &= \int \phi_k(\mathbf{x}_k)[\gamma_k(\mathbf{x}_k)+w_k]^T \aleph(\mathbf{x}^a_k;\hat{\mathbf{x}}^a_{k|k},\mathbf{P}^a_{k|k})d\mathbf{x}^a_k - \hat{\mathbf{x}}_{k+1|k}\hat{z}^T_{k|k},\end{aligned} \tag{10}$$

and

$$\begin{aligned}\mathbf{P}^{\mathbf{x}z}_{k+1|k} &= E[\tilde{\mathbf{x}}_{k+1|k}\tilde{z}^T_{k+1|k}|Y_k] \\ &= \int \mathbf{x}_{k+1}\gamma_{k+1}(\mathbf{x}_{k+1})^T \aleph(\mathbf{x}_{k+1};\hat{\mathbf{x}}_{k+1|k},\mathbf{P}_{k+1|k})d\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\hat{z}^T_{k+1|k}.\end{aligned} \tag{11}$$

The Kalman gain for state estimation is

$$K^{\mathbf{x}}_k = \mathbf{P}^{\mathbf{x}y}_{k+1|k}(\mathbf{P}^{yy}_{k+1|k})^{-1},$$

where

$$\mathbf{P}^{\mathbf{x}y}_{k+1|k} = (1-p_{k+1})\mathbf{P}^{\mathbf{x}z}_{k+1|k} + p_{k+1}\mathbf{P}^{\mathbf{x}z}_{k+1,k|k}.$$

Finally, the posterior state estimate and the posterior error covariance of the states are

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + K^{\mathbf{x}}_k(y_{k+1} - \hat{y}_{k+1|k}),$$

and

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - K^{\mathbf{x}}_k\mathbf{P}^{yy}_{k+1|k}(K^{\mathbf{x}}_k)^T.$$

The cross covariance between the measurement noise and the state is

$$\mathbf{P}^{\mathbf{x}w}_{k+1|k+1} = -K^{\mathbf{x}}_k\mathbf{P}^{yy}_{k+1|k}(K^w_k)^T.$$

Note-1 Under the assumption that the measurements are not delayed *i.e.*, $\beta_k = 0$, the delayed measurement equation reduces to the non-delayed measurement equation, and the formulated problem reduces to ordinary state estimation problem.

Note-2 The integrals appeared in equations (4) to (7), and (8) to (11) are generally intractable, hence these are approximated numerically. The accuracy of the filter depends on the accuracy of the numerical evaluation of the integrals.

As discussed earlier, the Gauss-Hermite quadrature rule is applied for approximating the intractable integrals.

3.4 Evaluation of multi-dimensional integral with Gauss quadrature rule

In this subsection, the evaluation of multi-dimensional integrals is described using the Gauss-Hermite quadrature rule. Although the Gauss-Hermite quadrature rule of integration is available in mathematics literature [27] for more than fifty years, the same has been incorporated in signal processing very recently, mainly due to the work of Ito and Xiong [14].

3.4.1 Single dimensional Gauss-Hermite quadrature rule

The single dimensional Gauss-Hermite quadrature rule is given by

$$\int_{-\infty}^{\infty} f(x) \frac{1}{(2\pi)^{1/2}} e^{-x^2} dx = \sum_{i=1}^{N} f(q_i) W_i,$$

where q_i and W_i represent the N quadrature points and the weights associated with them. To calculate the quadrature points and weights, let us consider a symmetric tridiagonal matrix J with zero diagonal elements and $J_{i,i+1} = \sqrt{i/2}; 1 \leq i \leq N-1$. The quadrature points are located at $\sqrt{2}x_i$, where x_i are the eigenvalues of J [11]. The weights W_i is the square of the first element of the i^{th} normalized vector.

The method of calculating quadrature points and the weights was first introduced by Golub *et al.* [28]. It works on the principle of moment matching. Instead of choosing the quadrature points arbitrarily in moment matching method, Golub's method can provide a deterministic way to select them with the best possible accuracy. Moment matching method is briefly outlined in the Appendix-A, for the sake of completeness.

3.4.2 Multidimensional extension of single dimensional Gauss-Hermite quadrature rule

Product rule:

Using product rule, the n dimensional integral given by

$$I_N = \int_{\Re^n} f(\mathbf{x}) \frac{1}{(2\pi)^{n/2}} e^{-(1/2)|\mathbf{x}|^2} d\mathbf{x} \tag{12}$$

could approximately be evaluated as

$$I_N = \sum_{i_1=1}^{N} ... \sum_{i_n=1}^{N} f(q_{i_1}, q_{i_2}, ..., q_{i_n}) W_{i_1} W_{i_2} ... W_{i_n} = \sum_{j=1}^{n_{s_p}} f(\xi_{p_j}) \omega_{p_j},$$

where $\xi_{p_j} = [q_{i_1}, q_{i_2}, ..., q_{i_n}]^T$ are multidimensional sample points, $\omega_{p_j} = W_{i_1} W_{i_2} ... W_{i_n}$ are corresponding weights and n_{s_p} is the number of sample points for the product rule. With this rule, the number of sample points *i.e.* n_{s_p} is equal to N^n.

In order to evaluate I_N for n^{th} order system, N^n number of quadrature points and weights are necessary. As an example, for a second order system and three point GHF, nine quadrature points and weights are required. Hence, the quadrature point requirement increases exponentially with the dimension of the system, *i.e.*, the method suffers from the *curse of dimensionality* problem. As a result, it can only be implemented for lower dimensional problems. To a very limited extent this could be overcome by ignoring the quadrature points on the diagonals because the weights associated with them are very small and hence they contribute negligibly to the computation of the integral. But with this crude way of approximation, the *curse of dimensionality* problem could not be overcome. To drastically reduce the support points requirement, the Smolyak rule [26] is introduced. It sharply reduces the computational burden and make the filter free from the *curse of dimensionality* problem.

Smolyak rule:

The *curse of dimensionality* problem associated with product rule could be circumvented by considerably reducing the number of support points using Smolyak formula [26]; also see [13] for its use in filtering. Using this rule, the product rule is replaced by a linear combination of tensor product. The Smolyak rule for approximating the n dimensional integral of a function $f(\mathbf{x})$ over a Gaussian pdf with accuracy level L, where L is a natural number, is described with the formula,

$$\begin{aligned} I_{n,L}(f) &= \int_{\Re^n} f(\mathbf{x})\aleph(\mathbf{x};\mathbf{0},\mathbf{I}_n) \\ &= \int_{\Re^n} f(\mathbf{x})\frac{1}{(2\pi)^{n/2}}e^{-(1/2)|\mathbf{x}|^2}d\mathbf{x} \\ &\approx \sum_{q=L-n}^{L-1}(-1)^{L-1-q}C_{n-1}^{L-1-q}\sum_{\Phi\in N_q^n}(I_{i_1}\otimes I_{i_2}\otimes ...\otimes I_{i_n})(f), \end{aligned} \tag{13}$$

where C is the binomial coefficient ($C_k^n = n!/k!(n-k)!$), $\otimes$ stands for the tensor product of I_{i_j} which is obtained from the univariate Gauss-Hermite quadrature rule of accuracy level $i_j \in \Phi$, where $\Phi \triangleq \{i_1, ..., i_n\}$ is an accuracy level set. The set N_q^n is defined as:

$$\begin{aligned} N_q^n &= \left\{\Phi : \sum_{j=1}^{n} i_j = n+q\right\} \qquad \text{for} \quad q \geq 0 (\text{where} \quad L-n \leq q \leq L-1) \\ &= \{\} \qquad\qquad\qquad\qquad \text{for} \quad q < 0 \end{aligned}$$

where $\{\}$ represents the null set. Accuracy level L implies that $I_{n,L}(f)$ is exact for the polynomials of the form $x_1^{i_1}x_2^{i_2}...\,x_n^{i_n}$ for $\sum_{j=1}^{n} i_j \leq (2L-1)$ [13]. I_{i_j} is single dimensional approximate of integral of function over Gaussian distribution, *i.e.*,

$$I_{i_j} = \int_{\Re} f(x)\aleph(x;0,1)dx = \sum_{x\in X_{i_j}} f(x)W_{i_j}(x) \tag{14}$$

where X_{i_j} is the univariate point set with accuracy level $i_j \in \Phi$, $j = 1, ..., n$. Combining (13) and (14), the integral $I_{n,L}(f)$ becomes,

$$I_{n,L}(f) \approx \sum_{q=L-n}^{L-1}\sum_{\Phi\in N_q^n}\sum_{x_1\in X_{i_1}} ... \sum_{x_n\in X_{i_n}} f(x_1, ..., x_n)\underbrace{(-1)^{L-1-q}C_{n-1}^{L-1-q}\prod W_{i_p}}_{weight},$$

where W_{i_p} is the weight associated with x_p. It must be noted that, for a grid point which appears more than one time in tensor product, the final weight of the point is the sum of the weights over all combinations of $X_{i_1} \otimes X_{i_2}, ..., X_{i_n}$ containing that point. The final set of the sparse-grid Gauss-Hermite (SGH) points and their corresponding weights are given by

$$\xi_{s_j} = X_{n,L} = \bigcup_{q=L-n}^{L-1}\bigcup_{\Phi\in\mathbf{N}_q^n}(X_{i_1}\otimes X_{i_2}\otimes ...\otimes X_{i_n})$$

and

$$\omega_{s_j} = (-1)^{L-1-q}C_{n-1}^{L-1-q}\prod W_{i_p}, \tag{15}$$

where $\bigcup$ is the union of the individual SGH points. Fig 1 illustrates the construction of sparse-grid Gauss-Hermite points for a two dimensional system with third degree of accuracy level ($L = 3$).

Hence, the integral of interest can be approximated as

$$I_{n,L}(f) \approx \sum_{j=1}^{n_{s_s}} f(\xi_{s_j})\omega_{s_j},$$

where n_{s_s} is the number of sample points used in Smolyak rule [13,26] for given accuracy level L and dimension n.

Note 1 The choice of the quadrature points are not unique. In [13], the single dimensional quadrature points (X_{i_j}) and weights are generated using the moment matching method. In this paper, we adopt the method described by Golub [28].

Note 2 The number of points in univariate X_{i_j} is higher than or equal to i_j. Similar to [13], here the number of elements in X_{i_j} is taken as $(2i_j - 1)$.

Note 3 The exactness of integration with the SGH points increases with the accuracy level L, however the computational burden also increases with it.

Note 4 The points and weights generated by unscented transform used in UKF-1RD are similar to the points and weights obtained from the Smolyak rule with $L = 2$ [13,26]. Also, Bin Jia and others [29] proved that the third-degree spherical-radial rule used in CKF-1RD can directly be obtained from the Smolyak rule under certain conditions. Hence, the existing UKF-1RD and CKR-1RD algorithms are special cases of SGHF-1RD.

4 Gauss-Hermite and sparse Gauss-Hermite filters for one step randomly delayed measurement

In this section, the Gauss-Hermite filter (GHF) and the sparse Gauss-Hermite filter (SGHF) are extended to deal with the one step randomly delayed measurement problems. To formulate a filter, the equations (4) to (7), and (8) to (11) are to be evaluated. The above mentioned equations contain an integral in the form,

$$I = \int_{\Re^n} f(\mathbf{x})\aleph(\mathbf{x};\mu,\Sigma)d\mathbf{x} = \int_{\Re^n} f(\mathbf{x})\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}e^{-(1/2)(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)}d\mathbf{x},$$

where μ and Σ are mean and covariance of $\mathbf{x}$. Unfortunately the integral mentioned above is intractable and hence can not be solved analytically for any arbitrary nonlinear function $f(\mathbf{x})$. In this paper, the integral is solved using multi dimensional Gauss-Hermite quadrature rule and multi dimensional sparse Gauss-Hermite quadrature rule.

4.1 GHF-1RD

To approximate the above integral, initially the Gauss-Hermite quadrature points (ξ_{p_j}) and weights (ω_{p_j}) have been generated using the product rule described in section 3.4.2. Next the points are transformed ($\chi_{p_j} = \sqrt{\Sigma}\xi_{p_j} + \mu$) depending on the mean and the covariance of Gaussian distribution over which the nonlinear function has to be integrated. Finally the integral is evaluated as: $I = \sum_j f(\sqrt{\Sigma}\xi_{p_j} + \mu)\omega_{p_j} = \sum_j f(\chi_{p_j})\omega_{p_j}$.

4.2 SGHF-1RD

In SGHF-1RD, the multidimensional support points (ξ_{s_j}) and weights (ω_{s_j}) are generated using Gauss-Hermite quadrature rule and the Smolyak formula [13,26] as mentioned in section 3.4.2. Similar to GHF-1RD case the points are transformed and integral is approximately evaluated as, $I = \sum_j f(\sqrt{\Sigma}\xi_{s_j} + \mu)\omega_{s_j} = \sum_j f(\chi_{s_j})\omega_{s_j}$.

4.3 Algorithm

The algorithm of the proposed GHF-1RD and SGHF-1RD is summarized as follows:

Step (i) Generation of Gauss-Hermite quadrature points and weights

- Generate univariate quadrature points and weight using Golub's technique.
- Combine them to obtain multidimensional Gauss-Hermite quadrature points and weights using product rule for GHF-1RD and Smolyak rule for SGHF-1RD.

Step (ii) Filter initialization

- Initialize the filter with the initial estimate $\hat{\mathbf{x}}^a_{0|0}$ and the initial error covariance $\mathbf{P}^a_{0|0}$, where $\hat{\mathbf{x}}^a_{0|0} = [\hat{\mathbf{x}}^T_0 \quad 0]^T$, and $\mathbf{P}^a_{0|0} = \text{diag}(\mathbf{P}_{0|0} \quad 0)$.
- Calculate the Gauss-Hermite quadrature points, ξ_j, and corresponding weights W_j for augmented state, where $j = 1, 2, ..., n^a_s$.
- Calculate another set of Gauss-Hermite quadrature points, ξ_i, and their corresponding weights W_i for non-augmented state, where $i = 1, 2, ..., n_s$.

Step (iii) Propagation of Gauss-Hermite quadrature points (predictor step)

- Perform the Cholesky decomposition of posterior error covariance of the augmented system:

$$\mathbf{P}^a_{k|k} = S^a_{k|k}(S^a_{k|k})^T$$

- Generate Gauss-Hermite quadrature points for the augmented system:

$$\chi_{j,k|k} = [(\chi^{\mathbf{x}}_{j,k|k})^T \quad (\chi^{w}_{j,k|k})^T]^T = \hat{\mathbf{x}}^a_{k|k} + S^a_{k|k}\xi_j$$

- Compute the time updated mean and covariance:

$$\hat{\mathbf{x}}_{k+1|k} = \sum_{j=1}^{n^a_s} W_j \phi_k(\chi^{\mathbf{x}}_{j,k|k})$$

$$\mathbf{P}_{k+1|k} = \sum_{j=1}^{n^a_s} W_j \phi_k(\chi^{\mathbf{x}}_{j,k|k})\phi_k(\chi^{\mathbf{x}}_{j,k|k})^T - \hat{\mathbf{x}}_{k+1|k}\hat{\mathbf{x}}^T_{k+1|k} + Q_k$$

Step (iv) Measurement noise estimate

- Perform the Cholesky decomposition of prior error covariance of state:

$$\mathbf{P}_{k+1|k} = S_{k+1|k}(S_{k+1|k})^T$$

- Generate Gauss-Hermite quadrature points around prior estimate of state:

$$\chi_{i,k+1|k} = \hat{\mathbf{x}}_{k+1|k} + S_{k+1|k}\xi_i$$

- Calculate the statistics of the measurements:

$$\hat{z}_{k+1|k} = \sum_{i=1}^{n_s} W_i \gamma_k(\chi_{i,k+1|k})$$

$$\mathbf{P}^{zz}_{k+1|k} = \sum_{i=1}^{n_s} W_i \gamma_k(\chi_{i,k+1|k})\gamma_k(\chi_{i,k+1|k})^T - \hat{z}_{k+1|k}\hat{z}^T_{k+1|k} + R_{k+1}$$

$$\hat{z}_{k|k} = \sum_{j=1}^{n_s^a} W_j[\gamma_k(\chi^{\mathbf{x}}_{j,k|k}) + \chi^w_{j,k|k}]$$

$$\mathbf{P}^{zz}_{k|k} = \sum_{j=1}^{n_s^a} W_j[\gamma_k(\chi^{\mathbf{x}}_{j,k|k}) + \chi^w_{j,k|k}][\gamma_k(\chi^{\mathbf{x}}_{j,k|k}) + \chi^w_{j,k|k}]^T - \hat{z}_{k|k}\hat{z}^T_{k|k}$$

- Mean and covariance of the randomly delayed measurement are given by:

$$\hat{y}_{k+1|k} = (1 - p_{k+1})\hat{z}_{k+1|k} + p_{k+1}\hat{z}_{k|k}$$

$$\mathbf{P}^{yy}_{k+1|k} = (1 - p_{k+1})\mathbf{P}^{zz}_{k+1|k} + p_{k+1}\mathbf{P}^{zz}_{k|k} + P_{k+1}(1 - p_{k+1})(\hat{z}_{k+1|k} - \hat{z}_{k|k})(\hat{z}_{k+1|k} - \hat{z}_{k|k})^T$$

- The Kalman gain for noise estimation is

$$(K^w_k) = \mathbf{P}^{wy}_{k+1|k}(\mathbf{P}^{yy}_{k+1|k})^{-1},$$

where

$$\mathbf{P}^{wy}_{k+1|k} = (1 - p_{k+1})R_{k+1}.$$

- Compute posterior estimate of measurement noise and it's error covariance:

$$\hat{w}_{k+1|k+1} = K^w_k(y_{k+1} - \hat{y}_{k+1|k})$$

$$\mathbf{P}^{ww}_{k+1|k+1} = R_{k+1} - K^w_k \mathbf{P}^{yy}_{k+1|k}(K^w_k)^T$$

Step (v) State estimation

- Calculate the cross covariances:

$$\mathbf{P}^{\mathbf{x}z}_{k+1,k|k} = \sum_{j=1}^{n_s^a} W_j \phi_k(\chi^{\mathbf{x}}_{j,k|k})[\gamma_k(\chi^{\mathbf{x}}_{j,k|k}) + \chi^{w}_{j,k|k}]^T - \hat{\mathbf{x}}_{k+1|k}\hat{z}^T_{k|k}$$

$$\mathbf{P}^{xz}_{k+1|k} = \sum_{i=1}^{n_s} W_i \chi^{\mathbf{x}}_{i,k+1|k}\gamma_k(\chi^{\mathbf{x}}_{i,k+1|k})^T - \hat{\mathbf{x}}_{k+1|k}\hat{z}^T_{k+1|k}$$

$$\mathbf{P}^{\mathbf{x}y}_{k+1|k} = (1-p_{k+1})\mathbf{P}^{\mathbf{x}z}_{k+1|k} + p_{k+1}\mathbf{P}^{\mathbf{x}z}_{k+1,k|k}$$

- Evaluate the Kalman gain for state estimate:

$$K^{\mathbf{x}}_k = \mathbf{P}^{\mathbf{x}y}_{k+1|k}(\mathbf{P}^{yy}_{k+1|k})^{-1}$$

- Compute the posterior state estimate and posterior error covariance of state:

$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + K^{\mathbf{x}}_k(y_{k+1} - \hat{y}_{k+1|k})$$

$$\mathbf{P}_{k+1|k+1} = \mathbf{P}_{k+1|k} - K^{\mathbf{x}}_k \mathbf{P}^{yy}_{k+1|k}(K^{\mathbf{x}}_k)^T$$

- The cross covariance between state and measurement noise is given by

$$\mathbf{P}^{w\mathbf{x}}_{k+1|k+1} = -K^{\mathbf{x}}_k \mathbf{P}^{yy}_{k+1|k}(K^{w}_k)^T.$$

The filtering algorithm is implemented recursively to determine the state $\hat{\mathbf{x}}_{k+1|k+1}$ at every time instant k. As discussed earlier, the accuracy of a filter depends on the accuracy of approximation of intractable integrals which appear in equations (4) to (7), and (8) to (11). UKF-1RD uses unscented transformation method and CKF-1RD uses spherical-radial rule to compute the integral. We propose to use the multidimensional Gauss-Hermite quadrature rule to approximate the same integrals. It is well established in literature that the multidimensional Gauss-Hermite quadrature method of integration provides better accuracy, compared to spherical radial rule of integration. Hence our proposed method is expected to perform with enhanced accuracy.

Remark 1. *The proposed method can also be extended for correlated noise case. If $S = E[q_k w_k]$, a pseudo-noise term $\bar{q}_k = q_k - SR^{-1}w_k$ can easily be shown to be uncorrelated with w_k. This can be used in our formulation, following the same steps as in [31] and [32] to compute the necessary covariance expressions and hence to get the approximate filter. Since the underlying procedure to 'de-correlate' the noise is identical for all the Gaussian filters (UKF, GHF and SGHF) which we compare our approximate filter with, we omit this extension in the simulation experiments in section 5.*

5 Simulation

In this section, a problem of maneuvering target tracking with constant but unknown turn rate has been considered. The same process model was used in our earlier publication [25] to demonstrate the performance of various quadrature filters when there was no delay in measurements. Here, we consider the measurements to be randomly delayed by up to one time step and use the newly proposed filters from section 4 to solve the filtering problem. The problems has been solved with the proposed delay filters. In this problem, the target is assumed to be maneuvering with constant turn rate. Such problems are popularly known as coordinated turn in avionics vocabulary [6]. The coordinated turn model, adopted for target motion is summarized in [30] and well described in [6].

5.1 Process model

To formulate the problem, we assume an object is maneuvering with a constant turn rate in a plane parallel to the ground *i.e.*, during maneuver the height of the vehicle remains constant. If the turn rate is a known constant, the process model remains linear. If the turn rate is constant but unknown, it can be inferred recursively from the range and bearing measurements using a filter. However, due to nonlinearity in measurements, the overall state space system becomes nonlinear. The equation of motion of an object in plane (x, y) following coordinated turn model can be described with the following equations:

$$\ddot{\mathrm{x}} = -\Omega\dot{\mathrm{y}},$$

$$\ddot{\mathrm{y}} = \Omega\dot{\mathrm{x}},$$

$$\dot{\Omega} = 0,$$

where x, and y represent the position in x, and y direction respectively. Ω is the angular rate which is a constant. State space representation of the above equations is

$$\dot{\mathbf{x}} = A\mathbf{x} + w,$$

where $\mathbf{x}$ is a state vector defined as $\mathbf{x} = [\mathrm{x}\ \dot{\mathrm{x}}\ \mathrm{y}\ \dot{\mathrm{y}}\ \Omega]^T$. The process noise is added to incorporate the uncertainties in process equation, arises due to wind speed, variation in turn rate, change in velocity *etc*. The target dynamics is discretized to obtain discrete process equation,

$$\mathbf{x}_{k+1} = F_k\mathbf{x}_k + w_k,$$

where

$$F_k = \begin{bmatrix} 1 & \dfrac{\sin(\Omega_{k-1}T)}{\Omega_{k-1}} & 0 & -\dfrac{1-\cos(\Omega_{k-1}T)}{\Omega_{k-1}} & 0 \\ 0 & \cos(\Omega_{k-1}T) & 0 & -\sin(\Omega_{k-1}T) & 0 \\ 0 & \dfrac{1-\cos(\Omega_{k-1}T)}{\Omega_{k-1}} & 1 & \dfrac{\sin(\Omega_{k-1}T)}{\Omega_{k-1}} & 0 \\ 0 & \sin(\Omega_{k-1}T) & 0 & \cos(\Omega_{k-1}T) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

5.2 Measurement model

In general, the nonlinear measurement equation could be written as

$$z_k = \gamma(\mathbf{x}_k) + v_k.$$

In this problem, we assume both the range and the bearing angle are available from measurements. So the nonlinear function $\gamma(.)$ becomes

$$\gamma(X_k) = \begin{bmatrix} \sqrt{\mathtt{x}_k^2 + \mathtt{y}_k^2} \\ atan2(\mathtt{y}_k, \mathtt{x}_k) \end{bmatrix} + v_k,$$

where *atan2* is the four quadrant inverse tangent function. Both w_k and v_k are white Gaussian noise of zero mean and Q and R covariance respectively, and T is sampling time. The process noise covariance, (Q), is given by

$$Q = q \begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} & 0 & 0 & 0 \\ \frac{T^2}{2} & T & 0 & 0 & 0 \\ 0 & 0 & \frac{T^3}{3} & \frac{T^2}{2} & 0 \\ 0 & 0 & \frac{T^2}{2} & T & 0 \\ 0 & 0 & 0 & 0 & 0.018T \end{bmatrix},$$

where $T = 0.5$ seconds and q is some constant given as $q = 0.1$. $R = \text{diag}([\sigma_r^2 \ \sigma_t^2])$ where $\sigma_r = 120$m and $\sigma_t = \sqrt{70}$mrad.

5.3 Simulation results

The truth state is initialized with $\mathbf{x}_0 = [1000\text{m} \ \ 30\text{m/s} \ \ 1000\text{m} \ \ 0\text{m/s} \ \ -3^\circ/s]^T$. The initial estimate $\hat{\mathbf{x}}_0$ is generated from a Gaussian distribution with mean $\mathbf{x}_0$ and covariance $\mathbf{P}_0 = \text{diag}([200\text{m}^2 \ \ 20\text{m}^2/\text{s}^2 \ \ 200\text{m}^2 \ \ 20\text{m}^2/\text{s}^2 \ \ 100\text{mrad}^2/\text{s}^2])$ respectively. The position, velocity and turn rate of the maneuvering target are estimated for 50 seconds using EKF-1RD, UKF-1RD, CKF-1RD, 3 points GHF-1RD, and SGHF-1RD (with accuracy level 3). The probability of $\beta_k = 1$ is taken as 0.5. To compare the performance of the above mentioned estimators, root mean square error (RMSE) of radial position, velocity and turn rate are calculated over 500 Monte Carlo runs. It was observed that EKF-1RD failed to track and its RMSE diverged in most of the Monte Carlo runs. Hence we exclude its results from the comparison.

The RMSEs of radial position, velocity and turn rate obtained from different filters are plotted in Fig 2. We observed that, during initial 7 seconds, the RMSEs obtained from the above mentioned filters are similar, hence we exclude that period from the plot. From the Fig 2, it could be seen that the RMSEs of proposed GHF-1RD, and SGHF-1RD are less compared to UKF-1RD and CKF-1RD. It indicates that the proposed methods provide better accuracy compared to their UKF and CKF counterparts.

To study the effect of probability of delay (p_k) on the estimation accuracy, in Fig 3, RMSEs of radial position, velocity and turn rate averaged over the time horizon are plotted against p_k. From the figure, it can be observed that for all the values of p_k, GHF-1RD and SGHF-1RD perform better compared to their UKF and CKF counterparts.

From Fig 2 and Fig 3, we observe that both the proposed GHF-1RD and SGHF-1RD are more accurate compared to the available delay filters. We also study the computational time of the above mentioned filters in Table 1. The relative differences in computational time, with the time for CKF-1RD taken as unity, are obtained on a personal computer with 64-bit operating system, 4 GB RAM and 3.33 GHz clock speed, on a MATLAB version 2010b. From the Table 1, we see that the execution time of GHF-1RD is quite high (almost 56 times of UKF-1RD and CKF-1RD). Despite the higher estimation accuracy, the high computational cost of GHF-1RD could restrict it from on board implementation. To decrease the computational load without affecting the accuracy, as we mentioned earlier, we proposed SGHF-1RD. The SGHF-1RD reduced the computational time of GHF-1RD more than 13 times without affecting the accuracy. In summary, we advocate to use SGHF-1RD, because it is capable of providing much accurate estimation with little increase in computational cost.

6 Discussions and conclusions

In this paper, two quadrature filtering algorithms, *viz.* the GHF and the SGHF are extended to estimate the states of a nonlinear system with one step randomly delayed measurements. The superiority of the proposed methods compared to its CKF and UKF counterparts has been demonstrated with the help of a maneuvering target tracking example. The proposed GHF-1RD provides higher accuracy in state estimation compared to UKF-1RD and CKF-1RD. However, the computational cost of GHF-1RD is very high and it suffers from the *curse of dimensionality* problem. To overcome the problem, we further formulated SGFH-1RD, which may be seen as the main contribution of this work. The accuracy of SGFH-1RD is comparable to GHF-1RD , while its computational load is significantly lower. Hence, we advocate to use SGHF-1RD due to its enhanced estimation accuracy and comparable computational load with existing methods.

Appendix-A: Moment matching method for generating quadrature points and weights

To evaluate a single dimensional integral with N number of quadrature points, $2N$ number of unknown parameters namely N points and N weights are required. A commonly used method is moment matching method [10, 12] which is given by:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_N \\ \vdots & \vdots & \ddots & \vdots \\ q_1^{N-1} & q_2^{N-1} & \cdots & q_N^{N-1} \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_N \end{pmatrix} = \begin{pmatrix} M_0 \\ M_1 \\ \vdots \\ M_{N-1} \end{pmatrix},$$

where M_j is the j^{th} moment, q_j is the j^{th} sample point and W_j is the weight associated with q_j. The unknown parameters are evaluated by solving the moment equation described above. However from the above equation $2N$ unknown parameters can not be evaluated. Some researchers advocate to select the quadrature points arbitrarily and calculate the corresponding weights by using moment equations [12], while others prefer to choose the quadrature points as the zeros of the Hermite polynomial [30], and determine the weights.

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Table 1: Relative computational time

Estimator	No. of Q points	Relative comp. time
CKF -1RD	14	1
UKF -1RD	15	1.015
GHF-1RD	2187	56.03
SGHF-1RD	127	4.412

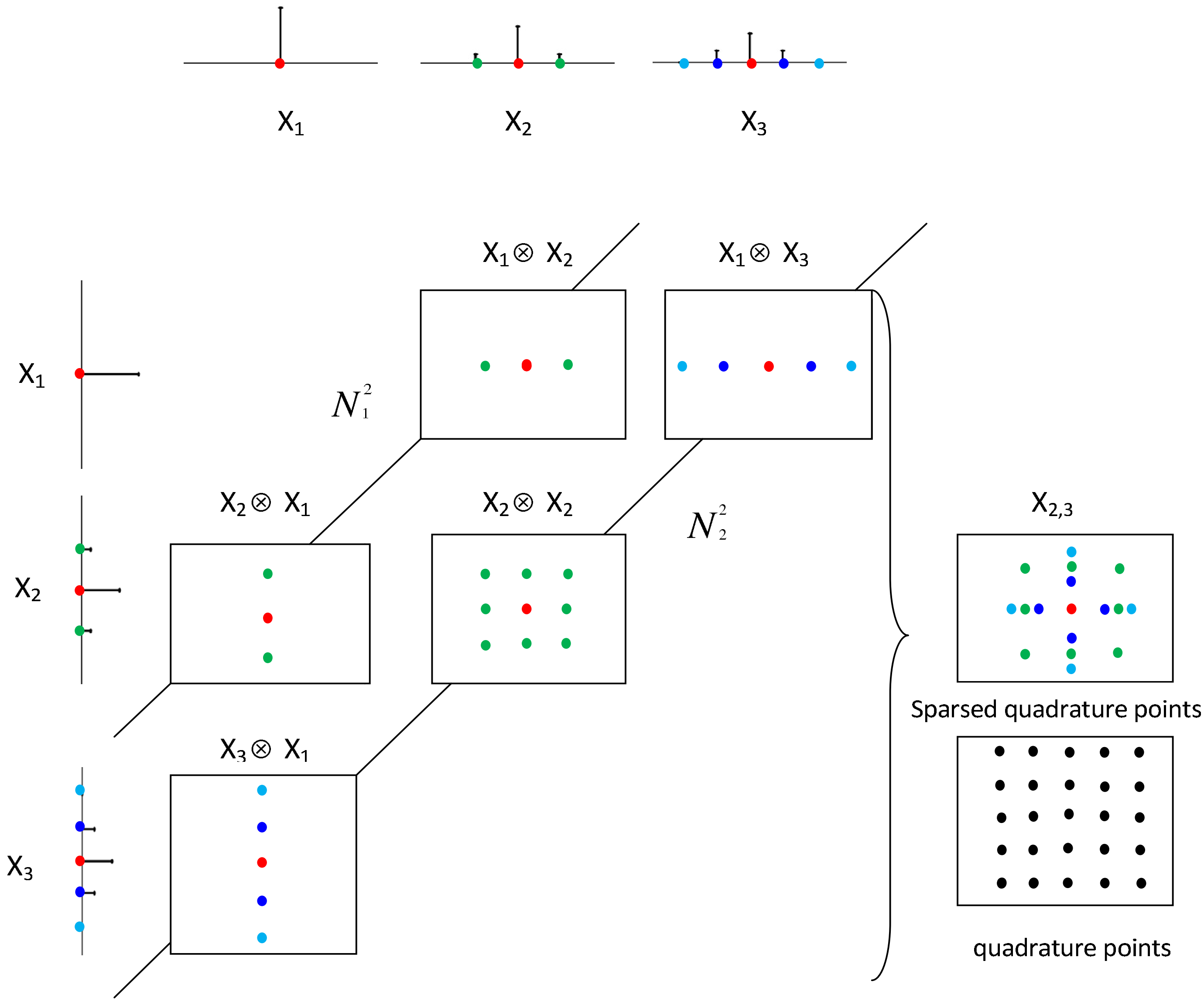

Figure 1: Generation of sparse-grid Gauss-Hermite points for $n = 2$, $L = 3$

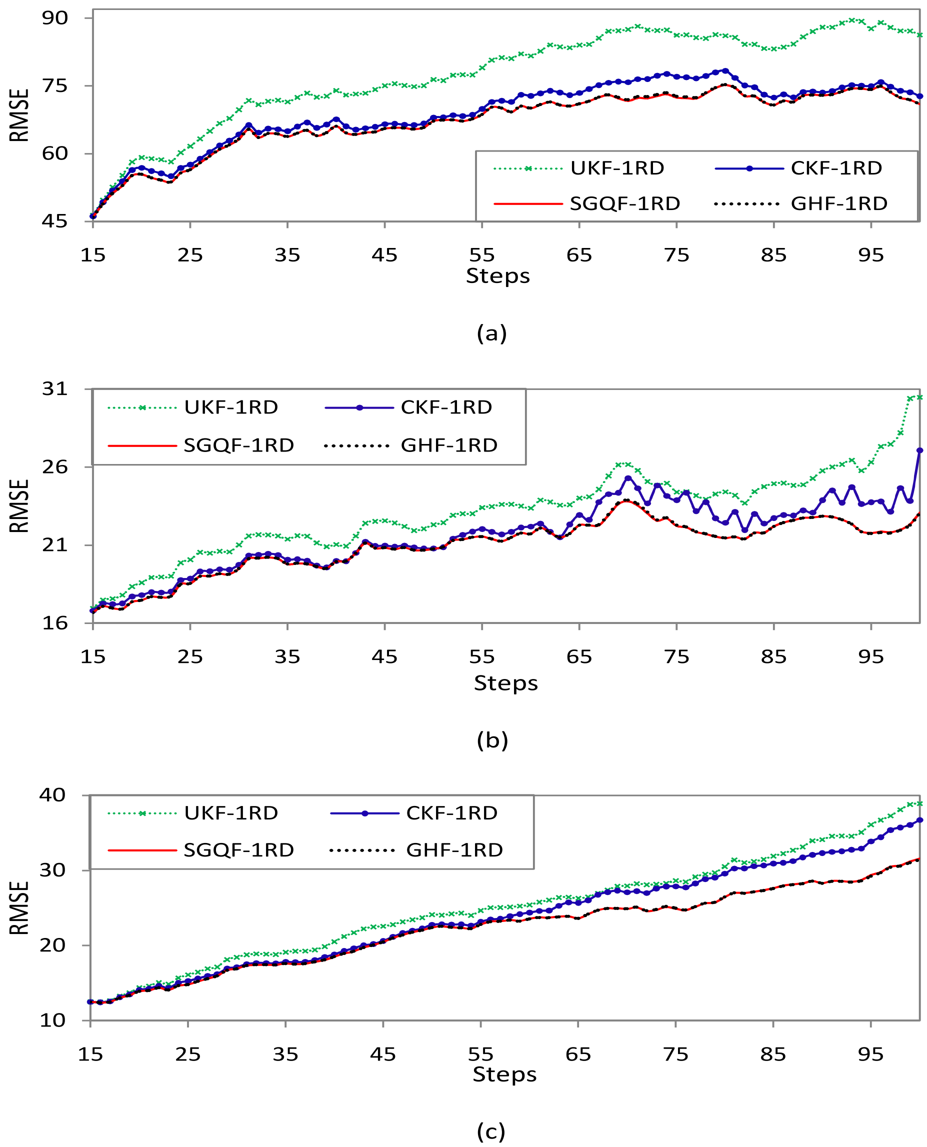

Figure 2: RMSE plot for $p = 0.5$: (a)position (b)velocity (c) turn rate in degree

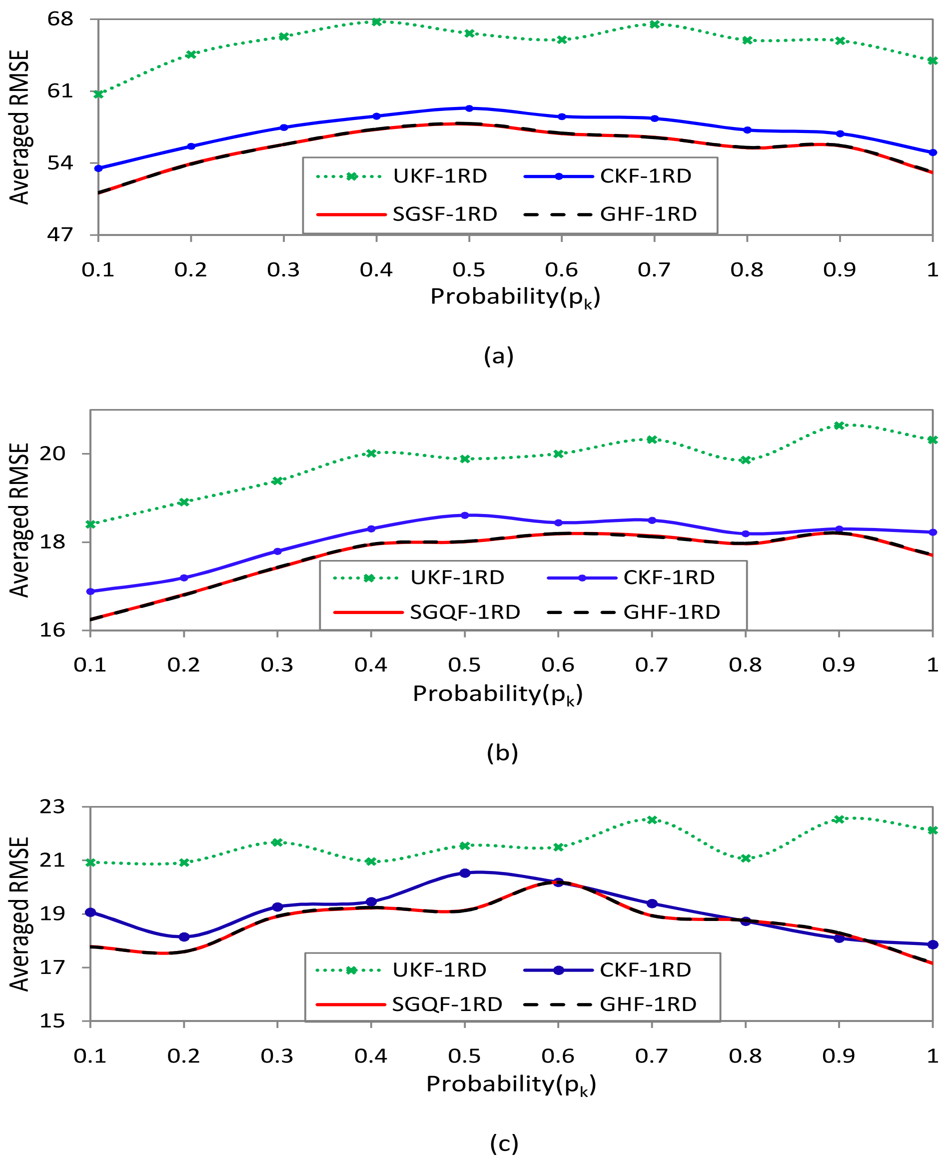

Figure 3: RMSE vs probability plot: (a)position (b)velocity (c) turn rate in degree