The Classification of Corank One Forced Symmetry Breaking

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1 Introduction

Suppose that a bifurcation equation changes its symmetry when a parameter varies. This is called a *forced-symmetry breaking* situation. It is to be contrasted with the phenomena of *spontaneous symmetry breaking* when the symmetry of the equations are kept constant but the symmetry of the bifurcating solutions can change. The simplest possible problem of that nature is of corank one:

$$f(x,\lambda,\mu) = x\, f_1(x^2,\lambda,\mu) + \mu f_2(x,\lambda,\mu). \tag{1.1}$$

The two parameter bifurcation problem f is $\mathbb{Z}_2$-equivariant when $\mu = 0$ but has no particular symmetries when $\mu \neq 0$.

In this work we would like to classify the simplest possible forced symmetry-breaking situation, that of corank one, to illustrate the power in such situation of the *Path Formulation* approach for bifurcation problems. Following the technical advances of Damon [4], Mond and Montaldi [15], Bridges and Furter [1] (for gradient problems) revitalised the Path Formulation for bifurcation problems which had been the starting point of the seminal paper of Golubitsky and Schaeffer [10].

A recent exposition of singularity theory applied to forced symmetry breaking bifurcation problems can be found in [8], following from the first papers using this approach [11, 5]. In [8] we classify corank two $(\mathbb{O}(2), \mathbf{1})$-bifurcation problems. The normal forms when $f_2(0) \neq 0$ are similar between the two theories, $(\mathbb{O}(2)$ or $\mathbb{Z}_2, \mathbf{1})$-symmetry breaking problems, but they obviously differ when $f_2(0) = 0$ because of the difference in coranks.

Path Formulation allows for a split of the discussion and calculations between the behaviour of the "core" of the problem, $f(x, 0)$, and the influence of the bifurcation parameters. Then it needs only a small adaptation to handle λ and μ as multidimensional parameters (or other more exotic parameter structures). Further studies of more complex examples are to be found in [9, 7] and their references.

2 Path formulation, a description

Let $f : (\mathbb{R}^{1+2}, 0) \to \mathbb{R}$ be a bifurcation problem, with $x \in \mathbb{R}$ being the state variable and $(\lambda, \mu) \in \mathbb{R}^2$ the bifurcation parameters. We assume that when $\mu = 0$ f is $\mathbb{Z}_2$-equivariant, i.e. $f(-x, \lambda, 0) =$

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$-f(x,\lambda,0)$, as when $\mu \neq 0$ f has no symmetries. this means that we can write f as

$$f(x,\lambda,\mu) = f_1(x,\lambda) + \mu f_2(x,\lambda,\mu). \tag{2.1}$$

The set $\vec{\mathcal{E}}^{\mathbb{Z}_2,\mathbf{1}}_{(x,\lambda,\mu)}$ of such bifurcation problems forms a module over the ring $\mathcal{E}^{\mathbb{Z}_2,\mathbf{1}}_{(x,\lambda,\mu)}$ of real valued functions $h : (\mathbb{R}^{1+2}, 0) \to \mathbb{R}$ such that $h(x,\lambda,\mu) = h_1(x^2,\lambda) + \mu h_2(x,\lambda,\mu)$. More precisely, let $u(x) = x^2$, we can show that $\vec{\mathcal{E}}^{\mathbb{Z}_2,\mathbf{1}}_{(x,\lambda,\mu)}$ and $\mathcal{E}^{\mathbb{Z}_2,\mathbf{1}}_{(x,\lambda,\mu)}$ have the following property.

Lemma 2.1. The sets $\vec{\mathcal{E}}^{\mathbb{Z}_2,\mathbf{1}}_{(x,\lambda,\mu)}$ and $\mathcal{E}^{\mathbb{Z}_2,\mathbf{1}}_{(x,\lambda,\mu)}$ are free $\mathcal{E}^{\mathbb{Z}_2}_{(x^2,\lambda,\mu)}$-modules generated by $\{x,\mu\}$ and $\{1,\mu x\}$, respectively.

Proof. Decomposition by the Division Theorem. $\square$

We consider the usual action of $\mathbb{Z}_2$ on $\mathbb{R}$ and deal with $\mathbb{Z}_2$-equivariant functions as in (1.1). The cores are $\epsilon_n u^n x$ with $u = x^2$, of $\mathcal{K}$-universal unfolding

$$F_n(x,\alpha) = \epsilon_n x^{2n+1} + \alpha_1 x^{2n-1} + \ldots + \alpha_{2n-1} x + \alpha_{2n},$$

with F_0^2 equal to

$$F_n^{Z_2}(x,\alpha) = (\epsilon_n u^n + \alpha_1 u^{n-1} + \ldots + \alpha_{2n-1})\, x.$$

We define $P_n(u,\beta) = \epsilon u^n + \sum_{i=1}^n \beta_i\, u^{n-i}$ and so $F_n^{Z_2}(x,\alpha) = P_n(x,(\alpha_1 \ldots \alpha_{2n-1}))\, x$.

Suppose that the *core* f_0 of f, $f_0(x) \overset{\text{def}}{=} f(x,0,0)$, is of *finite* codimension, that is, $f_0(x) = \epsilon_n x^n$ for $n \geq 2$ and $\epsilon_n^2 = 1$, and let F_0 be its $\mathcal{K}$-universal unfolding of parameters $\alpha \in \mathbb{R}^a$ where $a = \mathrm{cod}_{\mathcal{K}} f_0$. We consider f to be an unfolding of f_0 with l parameters and so,

From the Universal Unfolding Theorem, there exist changes of coordinates $T : (\mathbb{R}^{n+l}, 0) \to \mathrm{GL}_+(q,\mathbb{R})$, $X : (\mathbb{R}^{n+l}, 0) \to (\mathbb{R}^n, 0)$ and a map $p : (\mathbb{R}^l, 0) \to (\mathbb{R}^a, 0)$, called the *path* associated with f, such that

$$f(x,\lambda) = T(x,\lambda) F_0(X(x,\lambda), p(\lambda)). \tag{2.2}$$

The relation (2.2) between f_0 and p^*F_0 is a particular example of (restricted) contact-equivalence with distinguished parameter as introduced by Golubitsky-Schaeffer [12, 13].

In general, let $f, g \in \vec{\mathcal{E}}_{(x,\lambda)}$, f is *contact-equivalent* to g if

$$f(x,\lambda) = T(x,\lambda)\, g(X(x,\lambda), \Lambda(\lambda)) \tag{2.3}$$

for $T : (\mathbb{R}^{n+l}, 0) \to \mathrm{GL}_+(q,\mathbb{R})$, $X \in$ such that $X(0,0) = 0$, $X_x(0,0) \in \mathrm{GL}_+(n,\mathbb{R})$ and $\Lambda(0) = 0$, $\Lambda_\lambda(0) \in \mathrm{GL}_+(l,\mathbb{R})$. Note that Λ is now a diffeomorphism, on the contrary of p in (1.1). The set $\mathcal{K}_\lambda$ of contact equivalences (T, X, Λ) as in (2.3) has a group structure of semi-direct product.

The previous result (2.2) about paths representative of bifurcation germs is that f is $\mathcal{K}_\lambda$-contact-equivalent to p^*F_0 with equivalence $(T, X, \mathbf{I}_l)$. This last form of the problem has an obvious geometrical interpretation. The information we are really interested in, namely the zero-set, can be directly read off as the slice of the manifold $F_0^{-1}(0)$ over the path p.

The space of paths associated to F_n, $\vec{\mathcal{P}}^n_\lambda$, is composed of maps $(\Lambda, M) : (\mathbb{R}^{l_1+l_2}, 0) \to (\alpha_1 \ldots \alpha_{2n})$ such that Λ takes values only in the α_i's of odd ranks and M in α_i's of even ranks. Moreover, we assume that $\Lambda(\lambda) = \Lambda_1(\lambda_1) + \Lambda_2(\lambda_1,\lambda_2)$ and $\Lambda_2(\lambda_1, 0) = M(\lambda_1, 0) = 0$.

Therefore, instead of studying the action of $\mathcal{K}_\lambda$ on $\vec{\mathcal{E}}_{(x,\lambda)}$ we can study the space of paths associated with each particular core f_0, denoted by $\vec{\mathcal{P}}^{F_0}_\lambda$. The correct version of changes of coordinates for paths which corresponds to $\mathcal{K}_X$-equivalence for the associated diagrams is a contact-equivalence respecting the discriminant variety Δ_{F_0} associated to F_0.

More precisely, let $\pi_{F_0} : F_0^{-1}(0) \to \mathbb{R}^a$ be the restriction of the natural projection on the second element of (x, α). Let B_{F_0} be the local bifurcation set of F_0, that is,

$$B_{F_0} = \{ \ (x, \alpha) \mid F_0(x, \alpha) = 0 \ \text{ and } \ \mathrm{D}_x F_0(x, \alpha) \text{ is singular } \}.$$

Then, the discriminant variety of F_0 is given by $\Delta_{F_0} = \pi_{F_0}(B_{F_0})$. It is a variety of codimension 1 in $\mathbb{R}^a$ and indicates for which values of α there is a "bifurcation" (typically a fold point).

For our changes of coordinates on $\vec{\mathcal{P}}_\lambda^{F_0}$, we consider the group $\mathcal{K}_\Delta^{F_0}$ of Δ_{F_0}-preserving contact-equivalences (cf. [4] for the general theory of $\mathcal{K}_V$-equivalence). That is, $p, q \in \vec{\mathcal{P}}_\lambda^{F_0}$ are *path equivalent* (or $\mathcal{K}_\Delta^{F_0}$-equivalent) if

$$p(\lambda) = H(\lambda, q(\Lambda(\lambda))) \tag{2.4}$$

where

- $\Lambda : (\mathbb{R}^l, 0) \to (\mathbb{R}^l, 0)$ is an orientation preserving diffeomorphism,
- $H : (\mathbb{R}^{l+a}, 0) \to (\mathbb{R}^a, 0)$ is a λ-parametrised family of local diffeomorphism on $\mathbb{R}^a$ (that is, $H_q(0,0) \in \mathrm{GL}_+(a, \mathbb{R})$),
- we ask, moreover, that H preserves Δ_{F_0} in the sense that $H(\lambda, q) \in \Delta_{F_0}, \forall \lambda, \forall q \in \Delta_{F_0}$.

Recall that, contrary to $\mathcal{K}$-equivalence, we cannot assume in general that H is a linear map because Δ_{F_0} is in general a singular variety. Actually, it is usually even impossible to work with explicit such Hs. But the power of Path Formulation comes from the fact that the tangent spaces of $\mathcal{K}_\Delta^{F_0}$ can be effectively computed (in particular with the help of computer algebra). The extended tangent space $\mathcal{T}_e\mathcal{K}_\Delta^{F_0}(p)$ is equal to

$$p^*[\mathcal{E}_{(\lambda,\alpha)} \cdot \mathrm{Derlog}(\Delta_{F_0})] + \mathcal{E}_\lambda \cdot < p_\lambda > .$$

The module $\mathrm{Derlog}(\Delta_{F_0})$ represents the vector fields of $\mathbb{R}^a$ tangent to Δ_{F_0}. They can be computed independently of the path. Geometrically, the elements of $\mathrm{Derlog}(\Delta_{F_0})$ are the vector fields of $\mathbb{R}^a$ which can be lifted as vector fields of $\mathbb{R}^{n+a}$ tangent to the q-manifold $F_0^{-1}(0)$ (cf. [14]). This represents the connection with the $\mathcal{K}_X$-equivalence of p^*F_0.

It is well-known that the geometrical interpretation of singularity theory has to lie in the complex domain for a clear understanding of it. This is the same in our case, the underlaying algebra has a full geometrical interpretation only in the holomorphic situation. Because we are interested in the real situation and only in finitely determined germs, we shall use determinacy to complexify the story then use the geometrical ideas in the complex realm before, finally, come back by taking real slices of our results. So, all the objects we are going to consider are to be understood as real slices of the corresponding complex objects.

It is then shown that a problem is of finite codimension in the $\mathcal{K}_X$-theory if and only if its path representatives are of finite codimension in the $\mathcal{K}_\Delta^{F_0}$-theory (cf. [15]). In that case, the $\mathcal{K}_X$-theory for p^*F_0 mirrors the $\mathcal{K}_\Delta^{F_0}$-theory for p (cf. [9, 16]).

For our changes of coordinates on $\vec{\mathcal{P}}_\lambda^{F_0}$, we consider the group $\mathcal{K}_\Delta^{F_0}$ of contact equivalences preserving Δ_{F_0} and all its discriminants subsets when the relevant α's are 0. That is, $p, q \in \vec{\mathcal{P}}_\lambda^{F_0}$ are *path-equivalent* (or $\mathcal{K}_\Delta^{F_0}$-equivalent) if

$$p(\lambda) = H(\lambda, q(\Lambda(\lambda))) \tag{2.5}$$

where

- $\Lambda : (\mathbb{R}^l, 0) \to (\mathbb{R}^l, 0)$ is an orientation preserving diffeomorphism which preserves the splitting of $\mathbb{R}^l$ $(\Lambda(\lambda^i) = (\Lambda^i(\lambda^i, 0), 0))$,
- $H : (\mathbb{R}^{l+a}, 0) \to (\mathbb{R}^a, 0)$ is a λ-parametrised family of local diffeomorphism on $\mathbb{R}^a$ (that is, $H_q(0,0) \in \mathrm{GL}_+(a, \mathbb{R})$),
- we ask, moreover, not only that H preserves Δ_{F_0} but, also, that it preserves the discriminants $\Delta_{F_0^i}$ when $\alpha_{i+1} = \lambda_{i+1} = \ldots = \alpha_m = \lambda_m = 0$, for $i = 1 \ldots m$.

The (extended) tangent space at $p \in \vec{\mathcal{P}}_\lambda^{F_0}$ is given by

$$p^*[\mathcal{E}_{(\lambda,\alpha)} \cdot \mathrm{Derlog}(\Sigma)] + \mathcal{E}_\lambda \cdot < p_\lambda >,$$

and the unipotent tangent space is

$$p^*[\mathcal{E}_{(\lambda,\alpha)} \cdot \mathrm{UDerlog}(\Sigma)] + \mathcal{M}_\lambda^2 \cdot < p_\lambda >,$$

where $\mathrm{Derlog}(\Sigma)$ is the $\mathcal{E}_{(\lambda,\alpha)}$-module of vector fields tangent to Δ_{F_0}, which are also tangent to $\Delta_{F_0^i}$ when $\alpha_{i+1} = \lambda_{i+1} = \ldots = \alpha_m = \lambda_m = 0$, for $i = 1 \ldots m$, and $\mathrm{UDerlog}(\Sigma)$ consists of the vector fields whose linearisation at the origin is a nilpotent upper triangular matrix. We refer to [3] for the background theory and information on unipotent equivalence and its fundamental role in determinacy theory, in particular to estimate the higher order terms for p, that is, the terms in the Taylor series expansion we can discard for any member of the $\mathcal{K}_\Delta^{F_0}$-class of p.

2.1 Contact versus path equivalence

2.1.1 Contact equivalence

One can extend the Golubitsky-Schaeffer theory to deal with our set-up. The bifurcation germs belong to $\vec{\mathcal{E}}^{\,\Sigma}_{(x,\lambda)}$. We define $\mathcal{K}_\lambda^\Sigma$ as the group of contact-equivalences (T, X, Λ) acting on $\vec{\mathcal{E}}^{\,\Sigma}_{(x,\lambda)}$, respecting our different splittings in parameter space and symmetries of state space. Under our assumptions on Γ, the abstract theory of Damon [5] insures that $\mathcal{K}_\lambda^\Sigma$ is a well-defined geometric subgroup of $\mathcal{K}$ acting without restrictions on maps $h : \mathbb{R}^{n+l} \to \mathbb{R}^p$ and so that the usual results and techniques apply with respect to recognition, determinacy and unfolding theories.

Theorem 2.4. Let the Extension Hypothesis be satisfied.

(a) If $f \in \vec{\mathcal{E}}^{\,\Sigma}_{(x,\lambda)}$ has a core of finite $\mathcal{K}_\lambda$-codimension, there exists a path p such that f is $\mathcal{K}_\lambda$-equivalent to p^*F_0.

(b) $\mathrm{cod}_{\mathcal{K}_\Delta^{F_0}} p < \infty$ if and only if $\mathrm{cod}_{\mathcal{K}_\lambda} p^*F_0 < \infty$.

In that case, a map P is a $\mathcal{K}_\Delta^{F_0}$-universal unfolding of p if and only if P^*F_0 is a $\mathcal{K}_\lambda$-universal unfolding for p^*F_0.

(c) Let $p, \hat{p}$ be two paths in $\vec{\mathcal{P}}_\lambda^{F_0}$. Then, p is $\mathcal{K}_\Delta^{F_0}$-equivalent to $\hat{p}$ if and only if p^*F_0 is $\mathcal{K}_\lambda$-equivalent to $\hat{p}^*F_0$ for finite codimension problems.

To evaluate the set of higher order terms, traditionally denoted by $\mathcal{P}(p)$ for the path and $\mathcal{P}(f)$ for the bifurcation germs, we know that $\mathcal{P}(p)$, resp. $\mathcal{P}(p^*F_0)$, contain the largest intrinsic (that is, $\mathcal{K}_\Delta^{F_0}$, resp. $\mathcal{K}_\lambda$-invariant) subspace of $\mathcal{TUK}_\Delta^{F_0}(p)$, resp. $\mathcal{TUK}_\lambda(p^*F_0)$. This is often enough to get a good estimate, in particular at low codimensions. Thus, the following result is of use

$$\omega_p(\mathcal{TUK}_\Delta^{F_0}(p)) \subset \mathcal{TUK}_\lambda(p^*F_0).$$

3 $(\mathbb{Z}_2, 1)$-Symmetry Breaking

The group of changes of coordinates $\mathcal{K}_\Delta^{F_n}$ on $\vec{\mathcal{P}}_\lambda^n$ is the group of contact-equivalences (H, Λ) such that H preserves Δ_n and $\Delta_n^{\mathbb{Z}_2}$ (when $\mu = 0$), the discriminants of F_n and $F_n^{Z_2}$, respectively.

The (extended) tangent space at $\bar{\alpha} \in \mathcal{P}$ is

$$\bar{\alpha}^*\mathrm{Derlog}(\Sigma) + \mathcal{E}_{(\lambda,\mu)} < \bar{\alpha}_\lambda, \bar{\alpha}_\mu >,$$

and the unipotent tangent space is

$$\bar{\alpha}^*\mathrm{UDerlog}(\Sigma) + \mathcal{M}^2_{(\lambda,\mu)} < \bar{\alpha}_\lambda, \bar{\alpha}_\mu > .$$

We look now at the simplest situation when $l_1 = l_2 = 1$. To simplify the notation we use $\lambda = \lambda_1$ and $\mu = \lambda_2$. The paths are

$$\Lambda(\lambda, \mu) = \Lambda_1(\lambda) + \mu p(\lambda, \mu),$$

and

$$M(\lambda, \mu) = \mu q(\lambda, \mu).$$

3.1 Extension property

We split the unfolding parameters α of F_n between the odd and the even ranked terms: $\alpha = (\alpha^o, \alpha^e) = (\alpha_1, \alpha_3 \ldots, \alpha_2 \ldots)$. We define a $\mathbf{Z}_2$-action on the space of α via $(-1)\alpha = (\alpha^o, -\alpha^e)$ and we can check that $\mathbf{Z}_2$ acts on F_n in the following way:

$$F_n(-x, \alpha^o, -\alpha^e) = -F_n(x, \alpha^o, \alpha^e). \tag{3.1}$$

As a consequence, $F_n^{Z_2}$ is given by the restriction of F_n to Fix $\mathbf{Z}_2$.

We define $\Delta_n^{\mathbb{Z}_2}$ as the subset of $\alpha \in \mathbb{R}^{2n}$ where $F_n^{Z_2}(\cdot, \alpha)$ has a singular point. Explicitly, we can see that $\Delta_n^{\mathbb{Z}_2}$ is a reducible hypersurface formed from $\beta_n (= \alpha_{2n-1}) = 0$ and

$$\{ \ \beta \mid \exists\, u, \ P_n(u, \beta) = 0, \ (P_n)_u(u, \beta) = 0 \ \}.$$

The next proposition shows that this symmetric situation is perfectly induced from the general non symmetric case we saw previously. Note that the notations in [9] are slightly different.

Proposition 3.1 ([9]).

(a) Let $\{\xi_j\}_{j=1}^{2n}$ be the generators of Derlog(Δ_n)) as defined in (3.3). A free set of generators for Derlog($\Delta_n^{\mathbb{Z}_2}$) is provided by $\{ \xi_{2k-1}(\beta, 0)|_{\mathrm{Fix}\,\mathbf{Z}_2} \}_{k=1}^n$.

(b) A vector field ν is liftable over $(F_n^{Z_2})^{-1}(0)$ if and only if $\nu \in$ Derlog($\Delta_n^{\mathbb{Z}_2}$) if and only if it has an extension which is liftable over $F_n^{-1}(0)$.

Corollary 3.2. The Extension Property is satisfied by Δ_n and $\Delta_n^{\mathbb{Z}_2}$.

3.1.1 Calculations of the Derlogs

In [2] an explicit procedure to compute the elements of Derlog(Δ_n) is given. Let $\mathcal{J}(F_n)$ denote the ideal of $\mathcal{E}_{(x,\alpha)}$ generated by $(F_n)_x$. From the Preparation Theorem, the $\mathcal{E}_\alpha$-module $\mathcal{E}_{(x,\alpha)}/\mathcal{J}(F_n)$ is finitely generated by $\{x^{2n-i}\}_{i=1}^{2n}$. Therefore we can find $\{a_{ij}\}_{i,j=1}^{2n} \subset \mathcal{E}_\alpha$, such that

$$x^{j-1}F_n(x,\alpha) = \sum_{i=1}^{2n} a_{ij}(\alpha)\, x^{n-i} + p_j(x,\alpha)\,(F_n)_x(x,\alpha). \tag{3.2}$$

Then, for $j = 1 \ldots n$, the vector fields

$$\xi_j(\alpha) \stackrel{\text{def}}{=} (a_{1j}(\alpha), \ldots, a_{2nj}(\alpha)) \tag{3.3}$$

form a free set of generators for the $\mathcal{E}_\alpha$-module Derlog(Δ_n).

Let P_n be the invariant representative of F_n. The components of the generators for $\Delta_n^{\mathbb{Z}_2}$ are then given by

$$u^{j-1}P_n(u,\beta) = \sum_{i=1}^{n} b_{ij}(\beta)\, u^{n-i} + p_j(u,\beta)\,(P_n)_x(x,\beta)\, x. \tag{3.4}$$

Recall that $(P_n)_x(x,\beta)\, x \neq (P_n)_u(x^2,\beta)$. We see from (3.4) the restriction relation between $\{a_{ij}\}_{i,j=1}^{2n}$ and $\{b_{ij}\}_{i,j=1}^{n}$.

3.2 Classification

In this section we classify the two parameter problems up to topological codimension 2. It is clear that the topological $\mathcal{K}_\lambda^{\mathbb{Z}_2}$-codimension of f_1 is a lower bound for the topological codimension of f. Therefore we see that we need to consider only three cores: the cusp, F_2, the butterfly, F_3, and F_4 because they are the only cores who support a f_1 of $\mathcal{K}_\lambda^{\mathbb{Z}_2}$-topological codimension less or equal to 2.

Theorem 3.3. The list of problems of topological codimension up to 2 is given in the following table. We list the universal unfoldings, the germs are obtained by setting the unfolding parameters $\alpha = \beta = 0$. We associate I with the core F_2, II with the core F_3 and III with the core F_4. The letters differentiate between germs with different f_1, and the number is the topological codimension of the problem. Note that those three pieces of information are enough to classify our problems.

CASE	UNIVERSAL UNFOLDING	top-cod	cod
Ia(0)	$(\epsilon_1 u + \delta_1\lambda)\, x + \hat{\delta}_1\mu$	0	0
Ia(1)	$(\epsilon_1 u + \delta_1\lambda)\, x + \mu(\alpha + \hat{\delta}_3\mu)$	1	1
Ia(2)	$(\epsilon_1 u + \delta_1\lambda)\, x + \mu(\alpha + \beta\mu + n_1\lambda + a\lambda\mu + \hat{\delta}_6\mu^2)$	2	$3-4$
Ib(1)	$(\epsilon_1 u + \delta_2\lambda^2 + \alpha)\, x + \mu(\hat{\delta}_1 + ax)$	1	$1-2$
Ib(2)	$(\epsilon_1 u + \delta_2\lambda^2 + \alpha)\, x + \mu(\beta + \hat{\delta}_2 x + n_2\lambda + a\mu)$	2	$3-4$
Ic(2)	$(\epsilon_1 u + \delta_3\lambda^3 + \alpha + \beta\lambda)\, x + \mu(\hat{\delta}_1 + (a + b\lambda)\, x)$	2	$3-4$
IIa(1)	$(\epsilon_2 u^2 + \delta_1\lambda + \alpha u)\, x + \mu(\hat{\delta}_1 + ax^2 + (b + c\lambda)x^3)$	1	$2-4$
IIa(2)	$(\epsilon_2 u^2 + \delta_1\lambda + \alpha u)\, x + \mu(\beta + a\lambda + \hat{\delta}_3\mu + \hat{\delta}_4 x^2 + bx^3)$	2	$3-4$
IIb(2)	$(\epsilon_2 u^2 + m\lambda u + \delta_2\lambda^2 + \alpha + \beta u)\, x + \mu(\hat{\delta}_1 + dx + e_1 x^2 + e_2 x^3)$	2	$4-9$
IIIa(2)	$(\epsilon_3 u^3 + \delta_1\lambda + \alpha u + \beta u^2)\, x + \mu(\hat{\delta}_1 + dx^2 + e_1 x^3 + e_2 x^4 + e_3 x^5)$	2	$5-12$

The Greek letters ϵ_i, δ_i and $\hat{\delta}_i$ are normalised coefficients ± 1 (so they need to be non-zero). The letters a, b, c and d are modal parameters without any topological significance. Similarly, the letters e_i symbolise some modal polynomials in λ of no topological consequence. By contrast, the modal parameters m, n_1, n_2 need to avoid some special values, explicitly, $m^2 \neq 4\epsilon_2\delta_2$, $n_1^4 \neq 2\epsilon_1\delta_1$ and $n_2 \neq 0$.

Proof. In the next subsections we set-up and compute what we need to prove those results. □

Remark. Some of the germs in the list above belong to families:

Ia(l): $(\epsilon_1 u + \delta_1 \lambda)\, x + \mu\, (\hat{\delta}_2 \lambda + \hat{\delta} \mu^l)$,

I"m"($m-1$): $(\epsilon_1 u + \delta_m \lambda^m)\, x + \mu\, q(x, \lambda)$,

I$^{\rm n}$a($n-1$): $(\epsilon_n u^n + \delta_1 \lambda)\, x + \mu\, q(x, \lambda)$,

with $q(0,0) = \hat{\delta}_1 \neq 0$. Note that the function q has no influence on their topological types.

3.2.1 F$_2$-the cusp

We look first at the explicit calculations when $f_0(x) = x^3$. From the usual $\mathbb{Z}_2$-equivariant theory, we know that the only possible f_1 of finite codimension are of the form $f_1(u, \lambda) = \epsilon_1 u + \delta_m \lambda^m$, for some $m \geq 1$. The $\mathcal{K}^{\mathbb{Z}_2}$-universal unfolding F_2 is given by

$$F_2(x, \alpha_1, \alpha_2) = \epsilon_1 x^3 + \alpha_1 x + \alpha_2.$$

The discriminants are

$$\Delta_2 \equiv 4\epsilon_1 \alpha_1^3 + 27\alpha_2^2 = 0$$

and $\Delta_2^{\mathbb{Z}_2} \equiv \alpha_1 = 0$, so $\Sigma = (\Delta_2, \Delta_2^{\mathbb{Z}_2})$. The space of paths is given by

$$\vec{\mathcal{P}}^2_{(\lambda,\mu)} = \{\ (\alpha_1(\lambda,\mu), \alpha_2(\lambda,\mu)) \mid \alpha_1(\lambda,\mu) = \delta_1 \lambda^m + \mu p(\lambda,\mu)\,,\ \alpha_2(\lambda,\mu) = \mu q(\lambda,\mu)\ \}.$$

The module Derlog(Σ) is generated by

$$\begin{pmatrix} 2\alpha_1 \\ 3\alpha_2 \end{pmatrix}, \mu \begin{pmatrix} 9\alpha_2 \\ -2\epsilon_1 \alpha_1^2 \end{pmatrix}.$$

So the extended tangent space of a path is the following module

$$\mathcal{E}_{(\lambda,\mu)} < \begin{pmatrix} 2\delta_m \lambda^m + 2\mu p \\ 3\mu q \end{pmatrix}, \begin{pmatrix} 9\mu^2 q \\ -2\epsilon_1 \mu \alpha_1^2 \end{pmatrix}, \begin{pmatrix} \mu(m\delta_m \lambda^{m-1} + \mu p_\lambda) \\ \mu^2 q_\lambda \end{pmatrix}, \begin{pmatrix} \mu(p + \mu p_\mu) \\ \mu(q + \mu q_\mu) \end{pmatrix} >$$
$$+\mathcal{E}_\lambda < \begin{pmatrix} m\delta_m \lambda^{m-1} + \mu p_\lambda \\ \mu q_\lambda \end{pmatrix} > .$$

The unipotent tangent space is obtained by multiplying the first term by $\mathcal{M}_{(\lambda,\mu)}$ and the last term by $\mathcal{M}_\lambda^2$. The tangent space simplifies into

$$\mathcal{E}_\lambda < \begin{bmatrix} m\delta_m \lambda^{m-1} + \mu p_\lambda \\ \mu q_\lambda \end{bmatrix} > \tag{3.5}$$

$$+\mu \mathcal{E}_{(\lambda,\mu)} < \begin{bmatrix} 9\mu q \\ -2\epsilon_1 (\delta_m \lambda^m + \mu p)^2 \end{bmatrix}, \begin{bmatrix} 2mp - 2\lambda p_\lambda \\ 3mq - 2\lambda q_\lambda \end{bmatrix}, \begin{bmatrix} m\delta_m \lambda^{m-1} + \mu p_\lambda \\ \mu q_\lambda \end{bmatrix}, \begin{bmatrix} p + \mu p_\mu \\ q + \mu q_\mu \end{bmatrix} \tag{3.6}$$

The first term (3.5) takes care of the $\mathcal{E}_\lambda$-part of $\vec{\mathcal{P}}^2_{(\lambda,\mu)}$, so we need only to look at the μ-dependent part of $\vec{\mathcal{P}}^2_{(\lambda,\mu)}$ given by (3.6).

When $m = 1$ we can eliminate the first component of the path p using a change of variable in x and λ. Then, the third generator in (3.6) can be used to eliminate the first component in the tangent spaces. And so we are left with the following generators for the second component:

$$(-2\epsilon_1\lambda^2 - 9\delta_1\mu^2 q q_\lambda),\ (q + \mu q_\mu),\ (3q - 2\lambda q_\lambda),$$

from which we can conclude for Ia(i), $i = 1, 2$.

For $m = 2$ we have to consider only two cases. When $q(0) \neq 0$ we get the paths $p = a\mu$, $q = \hat{\delta}_1$ for a a modal parameter of no topological significance, as when $q(0) = 0$, we have to assume that $p = \hat{\delta}_2 \neq 0$ and $q = n_2\lambda + a\mu$ with $n_2 \neq 0$ (a has again no topological significance)

When $m = 3$, f_1 is of topological $\mathcal{K}^{\mathbb{Z}_2}_\lambda$-codimension two. We have just to check that the generic germ Ic(2), when $q^o \neq 0$, is indeed of topological codimension 2 for the Path Formulation.

3.2.2 $\mathbf{F_3}$-the butterfly

The next case corresponds to the core $f_0(x) = \epsilon_2 u^2$. The $\mathcal{K}^{\mathbb{Z}_2}$-universal unfolding F_3 is given by

$$F_3(x, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \epsilon_2 x^5 + \alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x + \alpha_4.$$

The discriminants are

$$\begin{aligned}\Delta_3 \equiv\ & 3125\alpha_4^4 + 256\epsilon\alpha_3^5 - 1600\epsilon\alpha_2\alpha_3^3\alpha_4 + 2250\epsilon\alpha_2^2\alpha_3\alpha_4^2 + 2000\epsilon\alpha_1\alpha_3^2\alpha_4^2 - 27\alpha_2^4\alpha_3^2\\ & -3750\epsilon\alpha_1\alpha_2\alpha_4^3 + 144\alpha_1\alpha_2^2\alpha_3^3 - 128\alpha_1^2\alpha_3^4 + 108\alpha_2^5\alpha_4 - 630\alpha_1\alpha_2^3\alpha_3\alpha_4\\ & +16\epsilon\alpha_1^4\alpha_3^3 + 825\alpha_1^2\alpha_2^2\alpha_4^4 - 900\alpha_1^3\alpha_3\alpha_4^2 - 4\epsilon\alpha_1^3\alpha_2^2\alpha_3^2 + 560\alpha_1^2\alpha_2\alpha_3^2\alpha_4\\ & +16\epsilon\alpha_1^3\alpha_2^3\alpha_4 - 72\epsilon\alpha_1^4\alpha_2\alpha_3\alpha_4 + 108\epsilon\alpha_1^5\alpha_4^2 = 0\end{aligned}$$

and

$$\Delta_3^{\mathbb{Z}_2} \equiv \alpha_3(\alpha_1^2 - 4\epsilon_2\alpha_3) = 0.$$

The space of paths is given by $\vec{\mathcal{P}}^3_{(\lambda,\mu)}$ is composed of maps $\alpha(\lambda,\mu)$ such that $\alpha_1(\lambda,\mu) = \alpha_{11}(\lambda) + \mu\, p_1(\lambda,\mu)$, $\alpha_2(\lambda,\mu) = \mu\, q_1(\lambda,\mu)$, $\alpha_3(\lambda,\mu) = \alpha_{31}(\lambda) + \mu\, p_2(\lambda,\mu)$ and $\alpha_4(\lambda,\mu) = \mu\, q_2(\lambda,\mu)$.

The paths for the three cases we look at are given by

IIa(1)	$(\mu(b + c\lambda), \mu a, \delta_1\lambda, \mu\hat{\delta}_1)$
IIa(2)	$(\mu b, \mu\hat{\delta}_2, \delta_1\lambda, \mu(a\lambda + \hat{\delta}_3\mu))$
IIb(2)	$(m\lambda + \mu e_2, \mu e_1, \delta_2\lambda^2 + \mu d, \mu\hat{\delta}_1)$

The module Derlog(Σ) is generated by

$$\begin{bmatrix} 2\alpha_1 \\ 3\alpha_2 \\ 4\alpha_3 \\ 5\alpha_4 \end{bmatrix}, \begin{bmatrix} 20\alpha_3 \\ 25\alpha_4 - 4\epsilon_2\alpha_1\alpha_2 \\ 10\epsilon_2\alpha_1\alpha_3 - 6\epsilon_2\alpha_2^2 \\ 15\epsilon_2\alpha_1\alpha_4 - 3\epsilon_2\alpha_2\alpha_3 \end{bmatrix}, \mu \begin{bmatrix} 15\alpha_2 \\ 20\alpha_3 - 6\epsilon_2\alpha_1^2 \\ 25\alpha_4 - 4\epsilon_2\alpha_1\alpha_2 \\ -2\epsilon_2\alpha_1\alpha_3 \end{bmatrix}, \mu \begin{bmatrix} 25\alpha_4 \\ -2\epsilon_2\alpha_1\alpha_3 \\ 15\epsilon_2\alpha_1\alpha_4 - 3\epsilon_2\alpha_2\alpha_3 \\ 10\epsilon_2\alpha_2\alpha_4 - 4\epsilon_2\alpha_3^2 \end{bmatrix}.$$

3.2.3 $\mathbf{F}_4$

In this case we do not need to do much because f_1 is already of topological $\mathcal{K}_\lambda^{\mathbb{Z}_2}$-codimension two. We have just to check that the generic germ IIIa(2), when $q^o \neq 0$, is indeed of topological codimension 2 for the Path Formulation.

The $\mathcal{K}^{\mathbb{Z}_2}$-universal unfolding F_4

$$F_4(x,\alpha) = \epsilon_3 x^7 + \alpha_1 x^5 + \alpha_2 x^4 + \alpha_3 x^3 + \alpha_4 x^2 + \alpha_5 x + \alpha_6.$$

The space of paths is given by $\vec{\mathcal{P}}^3_{(\lambda,\mu)}$ is composed of maps $\alpha(\lambda,\mu)$ such that $\alpha_1(\lambda,\mu) = \alpha_{11}(\lambda) + \mu\, p_1(\lambda,\mu)$, $\alpha_2(\lambda,\mu) = \mu\, q_1(\lambda,\mu)$, $\alpha_3(\lambda,\mu) = \alpha_{31}(\lambda) + \mu\, p_2(\lambda,\mu)$ $\alpha_4(\lambda,\mu) = \mu\, q_2(\lambda,\mu)$, $\alpha_5(\lambda,\mu) = \alpha_{51}(\lambda) + \mu\, p_3(\lambda,\mu)$ and $\alpha_6(\lambda,\mu) = \mu\, q_3(\lambda,\mu)$.

The generic path is given by $(\mu e_3, \mu e_2, \mu e_1, \mu d, \delta_1\lambda, \mu\hat{\delta}_1)$.

The module Derlog(Σ) is generated by

$$\begin{bmatrix} 2\alpha_1 \\ 3\alpha_2 \\ 4\alpha_3 \\ 5\alpha_4 \\ 6\alpha_5 \\ 7\alpha_6 \end{bmatrix}, \begin{bmatrix} 28\alpha_3 \\ 35\alpha_4 - 8\epsilon_3\alpha_1\alpha_2 \\ 42\alpha_5 + 14\epsilon_3\alpha_1\alpha_3 - 12\epsilon_3\alpha_2^2 \\ 49\alpha_6 + 21\epsilon_3\alpha_1\alpha_4 - 9\epsilon_3\alpha_2\alpha_3 \\ 28\epsilon_3\alpha_1\alpha_5 - 6\epsilon_3\alpha_2\alpha_4 \\ 35\epsilon_3\alpha_1\alpha_6 - 3\epsilon_3\alpha_2\alpha_5 \end{bmatrix}, \begin{bmatrix} 42\alpha_5 \\ 49\alpha_6 - 4\epsilon_3\alpha_1\alpha_4 \\ 28\epsilon_3\alpha_1\alpha_5 - 6\epsilon_3\alpha_2\alpha_4 \\ 21\epsilon_3\alpha_2\alpha_5 + 35\epsilon_3\alpha_1\alpha_6 - 8\epsilon_3\alpha_3\alpha_4 \\ 14\epsilon_3\alpha_3\alpha_5 + 28\epsilon_3\alpha_2\alpha_6 - 10\epsilon_3\alpha_4^2 \\ 21\epsilon_3\alpha_3\alpha_6 - 5\epsilon_3\alpha_4\alpha_5 \end{bmatrix}.$$

$$\mu\begin{bmatrix} 21\alpha_2 \\ 28\alpha_3 - 10\epsilon_3\alpha_1^2 \\ 35\alpha_4 - 8\epsilon_3\alpha_1\alpha_2 \\ 42\alpha_5 - 6\epsilon_3\alpha_1\alpha_3 \\ 49\alpha_6 - 4\epsilon_3\alpha_1\alpha_4 \\ -2\epsilon_3\alpha_1\alpha_5 \end{bmatrix}, \mu\begin{bmatrix} 35\alpha_4 \\ 42\alpha_5 - 6\epsilon_3\alpha_1\alpha_3 \\ 21\epsilon_3\alpha_1\alpha_4 + 49\alpha_6 - 9\epsilon_3\alpha_2\alpha_3 \\ 14\epsilon_3\alpha_2\alpha_4 + 28\epsilon_3\alpha_1\alpha_5 - 12\epsilon_3\alpha_3^2 \\ 21\epsilon_3\alpha_2\alpha_5 + 35\epsilon_3\alpha_1\alpha_6 - 8\epsilon_3\alpha_3\alpha_4 \\ 28\epsilon_3\alpha_2\alpha_6 - 4\epsilon_3\alpha_3\alpha_5 \end{bmatrix}, \mu\begin{bmatrix} 49\alpha_6 \\ -2\epsilon_3\alpha_1\alpha_5 \\ 35\epsilon_3\alpha_1\alpha_6 - 3\epsilon_3\alpha_2\alpha_5 \\ 28\epsilon_3\alpha_2\alpha_6 - 4\epsilon_3\alpha_3\alpha_5 \\ 21\epsilon_3\alpha_3\alpha_6 - 5\epsilon_3\alpha_4\alpha_5 \\ 14\epsilon_3\alpha_4\alpha_6 - 6\epsilon_3\alpha_5^2 \end{bmatrix}.$$

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